FRACTIONAL DERIVATIVES OF CONSTANT AND VARIABLE ORDERS APPLIED TO ANOMALOUS RELAXATION MODELS IN HEAT-TRANSFER PROBLEMS

by

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In this paper, we address a class of the fractional derivatives of constant and variable orders for the first time. Fractional-order relaxation equations of constants and variable orders in the sense of Caputo type are modeled from mathematical view of point. The comparative results of the anomalous relaxation among the various fractional derivatives are also given. They are very efficient in description of the complex phenomenon arising in heat transfer.

Key words: *heat transfer, fractional derivative of constant, fractional derivative of variable order, fractional differential equation, anomalous relaxation*

Introduction

Fractional derivatives (FDs) were utilized to model the complex phenomenon in science and engineering practice [1-7]. FDs of constant order, e.g., Riemann-Liouville and Caputo FDs, were considered to describe the anomalous relaxation [2,5,8-10]. The Riemann-Liouville type FD of variable-order was proposed in [11,12] and its extended versions were considered in [13]. The Caputo type FD of variable-order was developed in [14] and its extended versions were reported in [15,16].

Recently, FDs involving the kernels of the exponential and Mittag-Leffler functions were reported in [17-28]. For example, the FD with respect to the kernel of the exponential function in the sense of the Caputo type was firstly reported in [17-24] and the FD with the aid of the kernel of the exponential function in the sense of the Riemann-Liouville type was proposed in [25] and developed in [26]. The FD with the help of the kernel of the stretched exponential function in the sense of the

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Caputo type was presented in [27]. The FDs containing the Mittag-Leffler function kernels in the sense of the Caputo and Riemann-Liouville types were developed in [28].

Due to the above FDs in the different kernels, a class of the fractional derivatives of constant and variable orders, such as has not been reported. Motivated by the above ideas, the brief aims of the previous paper are to suggest the FD of constant order with respect to the Mittag-Leffler function and stretched Mittag-Leffler and exponential functions kernels in the sense of Gaussian-like type, the variable-order FDs in the Caputo-Fabrizio, Sun-Hao-Zhang- Baleanu and Atangana-Baleanu types as well as Mittag-Leffler and exponential functions and their stretched and generalized versions and to present their applications for handling the anomalous relaxation models in heat-transfer problems.

The paper is structured as follows. In Section 2, we present the concepts of the FDs of constant and variable orders. In Section 3, the models for the anomalous relaxation arising in heat-transfer problems are considered. Finally, the conclusion is outlined in Section 4.

FDs of constant and variable orders

Let $\Xi(\mu)$ be the differentiable function defined on the interval $[a,b]$, as well as $\Im(\omega)$ and $\Im(\omega(x))$ be normalization constants with respect to the orders ω $(0\le\omega\le1)$ and $\omega(x)$ $(0\le\omega(x)\le1)$, respectively. For more details of the normalization constants, readers refer to [17-28].

FDs of constant order

FD of the Caputo type with respect to the singular power-law kernel is given by [2,5,10]:

$$D_{a^+}^{(\omega)}\Xi(x)=\frac{1}{\Gamma(1-\omega)}\int_a^x\frac{1}{(x-\mu)^{\omega}}\left(\frac{d\Xi(\mu)}{d\mu}\right)d\mu, \tag{1a}$$

where $a\le x$, n is an integer, and $\Gamma(\cdot)$ denotes the Gamma function.

FD of the Caputo-Fabrizio type with respect to the exponential kernel, denoted by [17-23]

$$y=\varphi_1(x,\omega)=\exp\left(-\frac{\omega}{1-\omega}x\right)=\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^i\frac{x^i}{\Gamma(1+i)}, \tag{2a}$$

is given by:

$$^{CF}D_x^{(\omega)}\Xi(x)=\frac{(2-\omega)\Im(\omega)}{2(1-\omega)}\int_a^x\exp\left(-\frac{\omega}{1-\omega}(x-\mu)\right)\Xi^{(1)}(\mu)d\mu, \tag{2b}$$

where $a\le x$.

For $\Im(\omega)=2/(2-\omega)$, Eq.(2b) is given by [23]:

$$^{LNCF}_{\ *}D_x^{(\omega)}\Xi(x)=\frac{1}{1-\omega}\int_0^x\exp\left(-\frac{\omega}{1-\omega}(x-\mu)\right)\Xi^{(1)}(\mu)d\mu. \tag{3}$$

FD of the Sun-Hao-Zhang-Baleanu type with respect to the stretched exponential function, denoted by

$$y=\varphi_2(x,\omega)=\exp\left(-\frac{\omega}{1-\omega}x^{\omega}\right)=\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^i\frac{x^{i\omega}}{\Gamma(1+i)}, \tag{4a}$$

is given by [27]:

$${}^{SHZD}D_x^{(\omega)}\Xi(x)=\frac{\Im(\omega)}{(1-\omega)^{1/\omega}}\int_a^x\exp\left(-\frac{\omega}{1-\omega}(x-\mu)^{\omega}\right)\Xi^{(1)}(\mu)\,d\mu, \tag{4b}$$

where $a\le x$.

For simplicity, we have [27]

$$\Im(\omega)=\Gamma(\omega+1), \tag{5a}$$

such that

$${}^{SHZD}D_x^{(\omega)}\Xi(x)=\frac{\Gamma(1+\omega)}{(1-\omega)^{1/\omega}}\int_0^x\exp\left(-\frac{\omega}{1-\omega}(x-\mu)^{\omega}\right)\Xi^{(1)}(\mu)\,d\mu. \tag{5b}$$

FD with respect to the Mittag-Leffler function kernel, denoted by

$$y=\varphi_3(x,\omega)=\mathrm{E}_{\omega}\left(-\frac{\omega}{1-\omega}x\right)=\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^i\frac{x^{i}}{\Gamma(i\omega+1)}, \tag{6a}$$

is defined by:

$${}^{YC}D_x^{(\omega)}\Xi(x)=\frac{\Im(\omega)}{1-\omega}\int_a^x\mathrm{E}_{\omega}\left(-\frac{\omega}{1-\omega}(x-\mu)\right)\Xi^{(1)}(\mu)\,d\mu, \tag{6b}$$

where $a\le x$.

FD of the Atangana-Baleanu type with respect to the stretched Mittag-Leffler function kernel, denoted by

$$y=\varphi_4(x,\omega)=\mathrm{E}_{\omega}\left(-\frac{\omega}{1-\omega}x^{\omega}\right)=\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^i\frac{x^{i\omega}}{\Gamma(i\omega+1)}, \tag{7a}$$

is given by [28]:

$${}^{ABC}D_x^{(\omega)}\Xi(x)=\frac{\Im(\omega)}{1-\omega}\int_a^x\mathrm{E}_{\omega}\left(-\frac{\omega}{1-\omega}(x-\mu)^{\omega}\right)\Xi^{(1)}(\mu)\,d\mu, \tag{7b}$$

where $a\le x$.

FD of the Caputo-Fabrizio type with respect to the Gaussian-function kernel, denoted by

$$y=\varphi_5(x,\omega)=\exp\left(-\frac{\omega}{1-\omega}x^{2}\right)=\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^i\frac{x^{2i}}{\Gamma(1+i)}, \tag{8a}$$

is given by [24]:

$${}^{CF}D_x^{(\omega)}\Xi(x)=\frac{1+\omega^2}{\sqrt{\pi^{\omega}(1-\omega)}}\int_a^x\exp\left(-\frac{\omega}{1-\omega}(x-\mu)^{2}\right)\Xi^{(1)}(\mu)\,d\mu, \tag{8b}$$

where $a\le x$.

FD with respect to the stretched exponential function kernel in the sense of Gaussian-like type, denoted by

$$y=\varphi_6\left(x,\omega\right)=\exp\left(-\frac{\omega}{1-\omega}x^{2\omega}\right)=\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^{i}\frac{x^{2\omega i}}{\Gamma\left(1+i\right)}, \tag{9a}$$

is defined by:

$$^{GYC}D_x^{(\omega)}\Xi\left(x\right)=\frac{1+\omega^2}{\sqrt{\pi^{\omega}\left(1-\omega\right)}}\int_a^x \exp\left(-\frac{\omega}{1-\omega}\left(x-\mu\right)^{2\omega}\right)\Xi^{(1)}\left(\mu\right)d\mu, \tag{9b}$$

where $a\le x$.

FD with respect to the stretched Mittag-Leffler function kernel in the sense of Gaussian-like type, denoted by

$$y=\varphi_7\left(x,\omega\right)=\mathrm{E}_{\omega}\left(-\frac{\omega}{1-\omega}x^{2\omega}\right)=\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^{i}\frac{x^{2\omega i}}{\Gamma\left(1+i\omega\right)}, \tag{10a}$$

is defined by:

$$^{GYGC}D_x^{(\omega)}\Xi\left(x\right)=\frac{1+\omega^2}{\sqrt{\pi^{\omega}\left(1-\omega\right)}}\int_a^x \mathrm{E}_{\omega}\left(-\frac{\omega}{1-\omega}\left(x-\mu\right)^{2\omega}\right)\Xi^{(1)}\left(\mu\right)d\mu, \tag{10b}$$

where $a\le x$.

The LTs of the functions $x^{-1-\omega}/\Gamma\left(-\omega\right)$ and $\mathrm{E}_{\omega}\left(-x^{\omega}\right)$ are given by (see [2,12]):

$$L\left[\frac{x^{-1-\omega}}{\Gamma\left(-\omega\right)}\right]=s^{\omega}, \tag{11}$$

$$L\left[\mathrm{E}_{\omega}\left(-\beta x^{\omega}\right)\right]=\frac{s^{\omega-1}}{s^{\omega}+\beta}, \tag{12}$$

respectively, where L is the LT operator with respect to x, and β is a constant.

The LTs of the FDs are as follows:

$$L\left[{}^{SHZD}D_x^{(\omega)}\Xi\left(x\right)\right]=\frac{\Im\left(\omega\right)}{\left(1-\omega\right)^{1/\omega}}\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^{i}\frac{\Gamma\left(1+i\omega\right)}{\Gamma\left(1+i\right)}\frac{\left[s\hat{\Xi}\left(s\right)-\Xi\left(0\right)\right]}{s^{i\omega+1}}, \tag{13}$$

$$L\left[{}^{YC}D_x^{(\omega)}\Xi\left(x\right)\right]=\frac{\Im\left(\omega\right)}{1-\omega}\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^{i}\frac{\Gamma\left(i+1\right)}{\Gamma\left(i\omega+1\right)}\frac{\left[s\hat{\Xi}\left(s\right)-\Xi\left(0\right)\right]}{s^{i+1}}, \tag{14}$$

$$L\left[{}^{GYC}D_x^{(\omega)}\Xi\left(x\right)\right]=\frac{1+\omega^2}{\sqrt{\pi^{\omega}\left(1-\omega\right)}}\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^{i}\frac{\Gamma\left(1+2\omega i\right)}{\Gamma\left(1+i\right)}\frac{\left[s\hat{\Xi}\left(s\right)-\Xi\left(0\right)\right]}{s^{2i\omega+1}}, \tag{15}$$

$$L\left[{}^{GYGC}D_x^{(\omega)}\Xi\left(x\right)\right]=\frac{1+\omega^2}{\sqrt{\pi^{\omega}\left(1-\omega\right)}}\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^{i}\frac{\left[s\hat{\Xi}\left(s\right)-\Xi\left(0\right)\right]}{\Gamma\left(1+i\omega\right)s^{2i\omega+1}}. \tag{16}$$

For the details of the LTs of Eqs.(1a,2b,7b,8b) , see [2,17, 28].

A plot of the different kernels of the FDs for $\omega=0.6$ is displayed in Fig. 1.

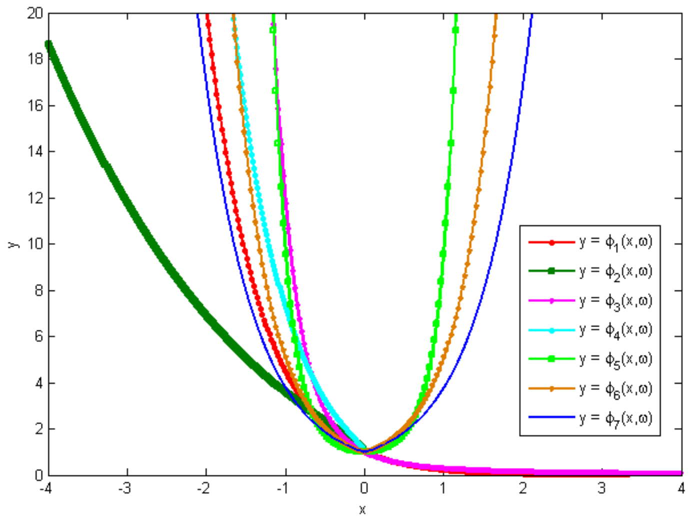

Fig. 1. The plot of the different kernels of the FDs for $\omega = 0.6$.

FDs of variable order

Variable-order FD of Caputo type in term of the singular power-law kernel is given by [15,16]:

$$D_{a^+}^{(\omega(x))}\Xi(x) = \frac{1}{\Gamma(1-\omega(x))}\int_a^x \frac{1}{(x-\mu)^{\omega(x)}}\left(\frac{d\Xi(\mu)}{d\mu}\right)d\mu,, \tag{17}$$

where $a \le x$ and n is an integer.

Variable-order FD of Caputo-Fabrizio type in term of the exponential kernel, denoted by

$$y = \varphi_1(x,\omega(t)) = \exp\left(-\frac{\omega(x)}{1-\omega(x)}x\right) = \sum_{i=0}^{\infty}\left(-\frac{\omega(x)}{1-\omega(x)}\right)^i \frac{x^i}{\Gamma(1+i)}, \tag{18a}$$

is defined by:

$$^{VYCF}D_x^{(\omega(x))}\Xi(x) = \frac{1}{1-\omega(x)}\int_a^x \exp\left(-\frac{\omega(x)}{1-\omega(x)}(x-\mu)\right)\Xi^{(1)}(\mu)d\mu, \tag{18b}$$

where $a \le x$.

Variable-order FD of the stretched exponential function in the sense of the Sun-Hao-Zhang-Baleanu type, denoted by

$$y = \varphi_2(x,\omega(t)) = \exp\left(-\frac{\omega(x)}{1-\omega(x)}x^{\omega(x)}\right) = \sum_{i=0}^{\infty}\left(-\frac{\omega(x)}{1-\omega(x)}\right)^i \frac{x^{i\omega(x)}}{\Gamma(1+i)}, \tag{19a}$$

is defined by:

$$^{VYSHZD}D_x^{(\omega)}\Xi(x) = \frac{\Im(\omega(x))}{1-\omega(x)}\int_a^x \exp\left(-\frac{\omega(x)}{1-\omega(x)}(x-\mu)^{\omega(x)}\right)\Xi^{(1)}(\mu)d\mu, \tag{19b}$$

where $a \le x$.

Generalized variable-order FD of the stretched exponential function, denoted by

$$y=\varphi_3\left(x,\omega(t)\right)=\exp\left(-x^{\omega(x)}\right)=\sum_{i=0}^{\infty}(-1)^i\frac{x^{i\omega(x)}}{\Gamma(i+1)}, \quad (20a)$$

is defined by:

$$^{GVYSHZD}D_x^{(\omega)}\Xi(x)=\frac{\Im\left(\omega(x)\right)}{1-\omega(x)}\int_a^x \exp\left(-(x-\mu)^{\omega(x)}\right)\Xi^{(1)}(\mu)\,d\mu, \quad (20b)$$

where $a \le x$.

Variable-order FD involving the Mittag-Leffler function kernel, denoted by

$$y=\varphi_4\left(x,\omega(t)\right)=\mathrm{E}_{\omega(x)}\left(-\frac{\omega(x)}{1-\omega(x)}x\right)=\sum_{i=0}^{\infty}\left(-\frac{\omega(x)}{1-\omega(x)}\right)^i\frac{x^i}{\Gamma\left(i\omega(x)+1\right)}, \quad (21a)$$

is given by:

$$^{VYGC}D_x^{(\omega(x))}\Xi(x)=\frac{1}{1-\omega(x)}\int_a^x \mathrm{E}_{\omega(x)}\left(-\frac{\omega(x)}{1-\omega(x)}(x-\mu)\right)\Xi^{(1)}(\mu)\,d\mu. \quad (21b)$$

Generalized variable-order FD involving the Mittag-Leffler function kernel, denoted by

$$y=\varphi_5\left(x,\omega(t)\right)=\mathrm{E}_{\omega(x)}(-x)=\sum_{i=0}^{\infty}(-1)^i\frac{x^i}{\Gamma\left(i\omega(x)+1\right)}, \quad (22a)$$

is given by:

$$^{GVYGC}D_x^{(\omega(x))}\Xi(x)=\frac{1}{1-\omega(x)}\int_a^x \mathrm{E}_{\omega(x)}\left(-(x-\mu)\right)\Xi^{(1)}(\mu)\,d\mu. \quad (22b)$$

The Atangana-Baleanu-type variable-order FD involving the stretched Mittag-Leffler function kernel, denoted by

$$y=\varphi_6\left(x,\omega(t)\right)=\mathrm{E}_{\omega(x)}\left(-\frac{\omega(x)}{1-\omega(x)}x^{\omega(x)}\right)=\sum_{i=0}^{\infty}\left(-\frac{\omega(x)}{1-\omega(x)}\right)^i\frac{x^{i\omega(x)}}{\Gamma\left(i\omega(x)+1\right)}, \quad (23a)$$

is defined by:

$$^{VYABC}D_x^{(\omega(x))}\Xi(x)=\frac{1}{\Gamma\left(1-\omega(x)\right)}\int_a^x \mathrm{E}_{\omega(x)}\left(-\frac{\omega(x)}{1-\omega(x)}(x-\mu)^{\omega(x)}\right)\Xi^{(1)}(\mu)\,d\mu, \quad (23b)$$

where $a \le x$.

Generalized variable-order FD involving the stretched Mittag-Leffler function kernel, denoted by

$$y=\varphi_7\left(x,\omega(t)\right)=\mathrm{E}_{\omega(x)}\left(-x^{\omega(x)}\right)=\sum_{i=0}^{\infty}(-1)^i\frac{x^{i\omega(x)}}{\Gamma\left(i\omega(x)+1\right)}, \quad (24a)$$

is defined by:

$$^{GVYABC}D_x^{(\omega(x))}\Xi(x)=\frac{1}{\Gamma\left(1-\omega(x)\right)}\int_a^x \mathrm{E}_{\omega(x)}\left(-(x-\mu)^{\omega(x)}\right)\Xi^{(1)}(\mu)\,d\mu, \quad (24b)$$

where $a \le x$.

The plot of the different kernels of the FDs for $\omega(x) = x - 1$ is displayed in Fig. 2.

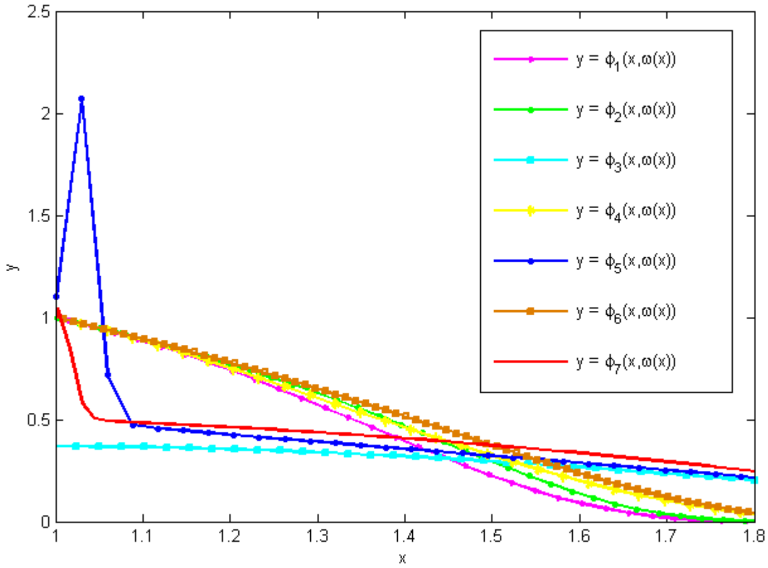

Fig. 2. The kernels of the different FDs for $\omega(x) = x - 1$.

The LT of the function $x^{-1-\omega(x)}/\Gamma(-\omega(x))$, given by Coimbra, takes the form [13]:

$$L\left[\frac{x^{-1-\omega(x)}}{\Gamma(-\omega(x))}\right] = s^{\omega(x)}. \tag{25}$$

The LT of the function $\mathrm{E}_{\omega(x)}\left(-x^{\omega(x)}\right)$ is given by [14]:

$$L\left[\mathrm{E}_{\omega(x)}\left(-\beta x^{\omega(x)}\right)\right] = \frac{s^{\omega(x)-1}}{s^{\omega(x)}+\beta}. \tag{26}$$

The LTs of the FDs of variable order are as follows:

$$L\left[{}^{VYCF}D_x^{(\omega(x))}\Xi(x)\right] = \frac{1}{1-\omega(x)}\sum_{i=0}^{\infty}\frac{\left(-\frac{\omega(x)}{1-\omega(x)}\right)^i\Gamma(i+1)\left[s\hat{\Xi}(s)-\Xi(0)\right]}{s^{i+1}}, \tag{27}$$

$$L\left[{}^{VYSHZD}D_x^{(\omega(x))}\Xi(x)\right] = \frac{\Im(\omega(x))}{1-\omega(x)}\sum_{i=0}^{\infty}\frac{\left(-\frac{\omega(x)}{1-\omega(x)}\right)^i\Gamma(i\omega(x)+1)\left[s\hat{\Xi}(s)-\Xi(0)\right]}{\Gamma(i+1)s^{i\omega(x)+1}}, \tag{28}$$

$$L\left[{}^{GVYSHZD}D_x^{(\omega(x))}\Xi(x)\right] = \frac{\Im(\omega(x))}{1-\omega(x)}\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(i\omega(x)+1)\left[s\hat{\Xi}(s)-\Xi(0)\right]}{\Gamma(i+1)s^{i\omega(x)+1}}, \tag{29}$$

$$L\left[{}^{VYGC}D_x^{(\omega(x))}\Xi(x)\right] = \frac{1}{1-\omega(x)}\sum_{i=0}^{\infty}\frac{\left(-\frac{\omega(x)}{1-\omega(x)}\right)^i\Gamma(i+1)\left[s\hat{\Xi}(s)-\Xi(0)\right]}{\Gamma(i\omega(x)+1)s^{i+1}}, \tag{30}$$

$$L\left[{}^{GVYGC}D_x^{(\omega(x))}\Xi(x)\right]=\frac{1}{1-\omega(x)}\sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma(i+1)\left[s\hat{\Xi}(s)-\Xi(0)\right]}{\Gamma(i\omega(x)+1)s^{i+1}}, \tag{31}$$

$$L\left[{}^{VYABC}D_x^{(\omega(x))}\Xi(x)\right]=\frac{1}{\Gamma(1-\omega(x))}\frac{s^{\omega(x)-1}\left[s\hat{\Xi}(s)-\Xi(0)\right]}{s^{\omega(x)}+\dfrac{\omega(x)}{1-\omega(x)}}, \tag{32}$$

$$L\left[{}^{GVYABC}D_x^{(\omega(x))}\Xi(x)\right]=\frac{1}{\Gamma(1-\omega(x))}\frac{s^{\omega(x)-1}\left[s\hat{\Xi}(s)-\Xi(0)\right]}{s^{\omega(x)}+1}. \tag{33}$$

The constant- and variable-order anomalous relaxation problems arising in heat transfer

The anomalous relaxation in the sense of the constant Caputo, Caputo-Fabrizio, Sun-Hao-Zhang- Baleanu types and variable-order FD of Caputo type were discussed in [15,23]. To present the temperature fields in the complex media, we consider a class of anomalous relaxation equations with the aid of the FD of constant and variable orders. The anomalous relaxation involving the variable-order FDs are in line with the result based on the generalized versions. Therefore, we consider the LT-type solutions for the anomalous relaxation with the aid of the generalized FDs of the variable order in this section.

The anomalous relaxation in the sense of the FD with respect to the Mittag-Leffler function kernel reads

$$-k\,{}^{YC}D_x^{(\omega)}\Xi_{YC}(x)=\Xi_{YC}(x), \tag{34a}$$

with the LT of the solution

$$\hat{\Xi}_{YC}(s)=\frac{\dfrac{k\Im(\omega)}{1-\omega}\displaystyle\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^i\frac{\Gamma(i+1)}{\Gamma(i\omega+1)}\frac{1}{s^{i+1}}}{\dfrac{k\Im(\omega)}{1-\omega}\displaystyle\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^i\frac{\Gamma(i+1)}{\Gamma(i\omega+1)}\frac{1}{s^{i}}+1}, \tag{34b}$$

where k is a constant, and $\Xi_{YC}(0)=1$ is the initial temperature field.

The anomalous relaxation involving the FD with respect to the stretched exponential function kernel in the sense of Gaussian-like type is

$$-k\,{}^{GYC}D_x^{(\omega)}\Xi_{GYC}(x)=\Xi_{GYC}(x), \tag{35a}$$

with the LT of the solution

$$\hat{\Xi}_{GYC}(s)=\frac{\dfrac{k(1+\omega^2)}{\sqrt{\pi^{\omega}}(1-\omega)}\displaystyle\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^i\frac{\Gamma(1+2\omega i)}{\Gamma(1+i)}\frac{1}{s^{2i\omega+1}}}{\dfrac{k(1+\omega^2)}{\sqrt{\pi^{\omega}}(1-\omega)}\displaystyle\sum_{i=0}^{\infty}\left(-\frac{\omega}{1-\omega}\right)^i\frac{\Gamma(1+2\omega i)}{\Gamma(1+i)}\frac{1}{s^{2i\omega}}+1}, \tag{35b}$$

where k is a constant, and $\Xi_{GYC}(0)=1$ is the initial temperature field.

The anomalous relaxation involving the FD with respect to the stretched Mittag-Leffler function kernel in the sense of Gaussian-like type can be written as

$$-k\,{}^{GYGC}D_x^{(\omega)}\Xi_{GYGC}(x)=\Xi_{GYGC}(x), \tag{36a}$$

with the LT of the solution

$$\hat{\Xi}_{GYGC}(s)=\frac{\dfrac{k\left(1+\omega^2\right)}{\sqrt{\pi^{\omega}(1-\omega)}}\sum_{i=0}^{\infty}\left(-\dfrac{\omega}{1-\omega}\right)^i\dfrac{1}{\Gamma(1+i\omega)}\dfrac{1}{s^{2i\omega+1}}}{\dfrac{k\left(1+\omega^2\right)}{\sqrt{\pi^{\omega}(1-\omega)}}\sum_{i=0}^{\infty}\left(-\dfrac{\omega}{1-\omega}\right)^i\dfrac{1}{\Gamma(1+i\omega)}\dfrac{1}{s^{2i\omega}}+1}, \tag{36b}$$

where k is a constant, and $\Xi_{GYGC}(0)=1$ is the initial temperature field.

The anomalous relaxation involving the variable-order FD of Caputo-Fabrizio type in term of the exponential kernel is:

$$-k\,{}^{VYCF}D_x^{(\omega(x))}\Xi_{VYCF}(x)=\Xi_{VYCF}(x), \tag{37a}$$

with the LT of the solution

$$\hat{\Xi}_{VYCF}(s)=\frac{\dfrac{k}{1-\omega(x)}\sum_{i=0}^{\infty}\left(-\dfrac{\omega(x)}{1-\omega(x)}\right)^i\dfrac{\Gamma(i+1)}{s^{i+1}}}{\dfrac{k}{1-\omega(x)}\sum_{i=0}^{\infty}\left(-\dfrac{\omega(x)}{1-\omega(x)}\right)^i\dfrac{\Gamma(i+1)}{s^{i}}+1}, \tag{37b}$$

where k is a constant, and $\Xi_{VYCF}(0)=1$ is the initial temperature field.

The anomalous relaxation in the sense of the variable-order FD of Caputo-Fabrizio type in term of the exponential kernel is represented as:

$$-k\,{}^{GVYSHZD}D_x^{(\omega(x))}\Xi_{GVYSHZD}(x)=\Xi_{GVYSHZD}(x), \tag{38a}$$

with the LT of the solution

$$\hat{\Xi}_{GVYSHZD}(s)=\frac{\dfrac{k\Im(\omega(x))}{1-\omega(x)}\sum_{i=0}^{\infty}\dfrac{(-1)^i\,\Gamma(i\omega(x)+1)}{\Gamma(i+1)\,s^{i\omega(x)+1}}}{\dfrac{k\Im(\omega(x))}{1-\omega(x)}\sum_{i=0}^{\infty}\dfrac{(-1)^i\,\Gamma(i\omega(x)+1)}{\Gamma(i+1)\,s^{i\omega(x)}}+1}, \tag{38b}$$

where k is a constant, and $\Xi_{GYGC}(0)=1$ is the initial temperature field.

The anomalous relaxation involving the FD with respect to the stretched Mittag-Leffler function kernel in the sense of Gaussian-like type is considered as:

$$-k\,{}^{GVYGC}D_x^{(\omega(x))}\Xi_{GVYGC}(x)=\Xi_{GVYGC}(x), \tag{39a}$$

with the LT of the solution

$$\hat{\Xi}_{GVYGC}(s)=\frac{\dfrac{k}{1-\omega(x)}\displaystyle\sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma(i+1)}{\Gamma(i\omega(x)+1)s^{i+1}}}{\dfrac{k}{1-\omega(x)}\displaystyle\sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma(i+1)}{\Gamma(i\omega(x)+1)s^{i}}+1}, \tag{39b}$$

where k is a constant, and $\Xi_{GVYGC}(0)=1$ is the initial temperature field.

The anomalous relaxation in the sense of the variable-order FD of Caputo-Fabrizio type in term of the exponential kernel is represented as:

$$-k\,{}^{GVYABC}D_x^{(\omega(x))}\Xi_{GVYABC}(x)=\Xi_{GVYABC}(x), \tag{40a}$$

with the LT of the solution

$$\hat{\Xi}_{GVYABC}(s)=\frac{\dfrac{k}{\Gamma(1-\omega(x))}\dfrac{s^{\omega(x)-1}}{s^{\omega(x)}+1}}{\dfrac{k}{\Gamma(1-\omega(x))}\dfrac{s^{\omega(x)}}{s^{\omega(x)}+1}+1}, \tag{40b}$$

where k is a constant, and $\Xi_{GVYABC}(0)=1$ is the initial temperature field.

Conclusion

In this work, we suggested a class of the FDs of constant and variable orders for the first time. The plots of the kernel functions of the different patterns in were discussed in detail. The LT-type solutions for the anomalous relaxations involving the FDs of constant and variable orders were also given. The proposed formulas are useful to open up the new prospects of describing the fractional-order heat-transfer equations in the complex media.

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Nomenclature

x -space coordinate, [m] κ – a coefficient, [m]
ω – a constant order, [-] $\omega(x)$ –a variable order, [-]
$\Xi(x)$ –the temperature fields in the different FDs, [K]

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