

EXACT TRAVELING-WAVE SOLUTIONS FOR LINEAR AND NONLINEAR HEAT-TRANSFER EQUATIONS

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The exact traveling-wave solutions for the linear and nonlinear heat-transfer equations at several different excess temperatures are addressed and investigated in this paper.

Key words: *heat-transfer equations, exact solution, travelling-wave transformation, excess temperatures*

1. Introduction

Ordinary differential equations (ODEs) and partial differential equations (PDEs) were used to describe the thermal problems in engineering sciences (see [1] and several related earlier references which are cited therein). Especially in heat-transfer problems, the PDEs [2] were adopted to govern the excess temperature fields in materials. In recent years, many different techniques were developed to derive the exact solutions for the heat-transfer equations, such as the Tanh method (TM) [3], Exp-function method (EM) [4], (G'/G)-expansion method (GEM) [5], heat-balance integral method [6], traveling-wave transformation method (TTM) [7,8] and other methods (see, e.g., [9-12]).

However, the traveling-wave solutions of the heat-transfer problems at several different excess temperatures have not yet been investigated. Motivated by the above investigations, the aim of the present paper is to propose the traveling-wave solutions for the linear and nonlinear heat-transfer equations.

The structure of the present paper is designed as follows. In Section 2, the TTM, used in this paper, is introduced. In Section 3, the traveling-wave solutions for the heat-transfer problems are proposed. Finally, the conclusion is presented in Section 4.

2. The method applied

In order to introduce the concept of the traveling-wave solution, we consider the following PDE with respect to ξ and τ :

$$\aleph \left(\frac{\partial \Theta(\xi, \tau)}{\partial \tau}, \frac{\partial^2 \Theta(\xi, \tau)}{\partial \xi^2}, \Theta^n(\xi, \tau) \right) = 0, \quad (1)$$

where n is a positive integer.

Following the argument in Ref. [7,8], we set up the TTM, which is given by:

$$\omega = \xi - \gamma\tau, \quad (2)$$

where γ is a constant.

With the aid of the following chain rules:

$$\frac{\partial \Theta(\xi, \tau)}{\partial \tau} = -\gamma \frac{\partial \Theta(\omega)}{\partial \omega} \quad (3)$$

and

$$\frac{\partial^2 \Theta(\xi, \tau)}{\partial \xi^2} = \frac{\partial^2 \Theta(\omega)}{\partial \omega^2}, \quad (4)$$

Eq. (1) can be transformed into the ODE with respect to ω , which is given by:

$$\aleph \left(-\gamma \frac{d\Theta(\omega)}{d\omega}, \frac{d^2\Theta(\omega)}{d\omega^2}, \Theta^n(\omega) \right) = 0. \quad (5)$$

After obtaining the solution of Eq. (5) by using the mathematical software, if we substitute Eq. (2) into the obtained solution, we get the traveling-wave solution.

3. Traveling-wave solutions for linear and nonlinear heat-transfer problems

At first, we consider the linear heat-transfer equation at the low excess temperature as follows (see [2]):

$$\frac{\partial \Xi(\xi, \tau)}{\partial \tau} = \alpha \frac{\partial^2 \Xi(\xi, \tau)}{\partial \xi^2} - \beta \Xi(\xi, \tau), \quad (6)$$

where α is the heat-diffusion coefficient, β is a constant and $\Xi(\xi, \tau)$ is the excess temperature.

Following Eq. (2), Eq. (6) can be written as follows:

$$-\gamma \frac{d\Xi(\omega)}{d\omega} = \alpha \frac{d^2\Xi(\omega)}{d\omega^2} - \beta \Xi(\omega). \quad (7)$$

With the help of the integrating-factor method [13] or the MatLab software, the exact solution of Eq. (7) is given by (see [13]):

$$\Xi(\omega) = \begin{cases} \varpi_1 e^{-k_1 \omega} + \varpi_2 e^{-k_2 \omega}, & \gamma^2 + 4\alpha\beta > 0, \\ (\varpi_3 + \varpi_4 \omega) e^{-\lambda \omega / 2}, & \gamma^2 + 4\alpha\beta = 0, \\ e^{-\lambda \omega / 2} (\varpi_5 \cos \varphi \omega + \varpi_6 \sin \varphi \omega), & \gamma^2 + 4\alpha\beta < 0, \end{cases} \quad (8)$$

where $\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5$ and ϖ_6 are constants, $\lambda = -\gamma / \alpha$, $k_1 = (-\gamma + \sqrt{\gamma^2 + 4\alpha\beta}) / 2\alpha$, $k_2 = (-\gamma - \sqrt{\gamma^2 + 4\alpha\beta}) / 2\alpha$ and $\varphi = \sqrt{-(\gamma^2 + 4\alpha\beta)} / 2\alpha$.

Substituting Eq. (2) into Eq. (8), we obtain

$$\Xi(\xi, \tau) = \begin{cases} \varpi_1 e^{-k_1(\xi-\gamma\tau)} + \varpi_2 e^{-k_2(\xi-\gamma\tau)}, & \gamma^2 + 4\alpha\beta > 0, \\ [\varpi_3 + \varpi_4(\xi - \gamma\tau)] e^{-\lambda(\xi-\gamma\tau)/2}, & \gamma^2 + 4\alpha\beta = 0, \\ e^{-\lambda(\xi-\gamma\tau)/2} [\varpi_5 \cos \varphi(\xi - \gamma\tau) + \varpi_6 \sin \varphi(\xi - \gamma\tau)], & \gamma^2 + 4\alpha\beta < 0, \end{cases} \quad (9)$$

where $\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5$ and ϖ_6 are constants, $\lambda = -\gamma / \alpha$, $k_1 = (-\gamma + \sqrt{\gamma^2 + 4\alpha\beta}) / 2\alpha$, $k_2 = (-\gamma - \sqrt{\gamma^2 + 4\alpha\beta}) / 2\alpha$ and $\varphi = \sqrt{-(\gamma^2 + 4\alpha\beta)} / 2\alpha$.

The graphs of the traveling-wave solutions in Eq. (6) are illustrated in Figs. 1(a), 1(b) and 1(c).

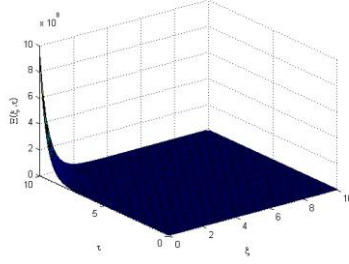


Fig.1(a). The traveling-wave solution for the linear heat-transfer equation (6) for $\gamma^2 + 4\alpha\beta > 0$.

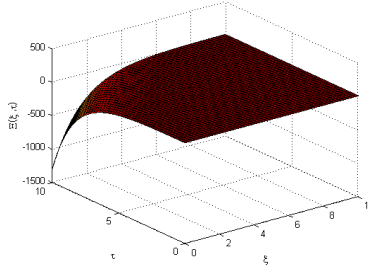


Fig. 1(b). The traveling-wave solution for the linear heat-transfer equation (6) for $\gamma^2 + 4\alpha\beta = 0$.

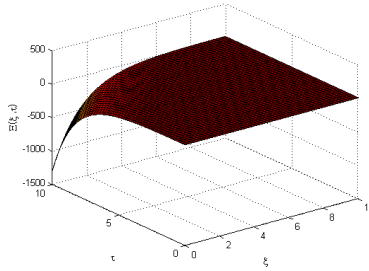


Fig.1(c). The traveling-wave solution for the linear heat-transfer equation (6) for $\gamma^2 + 4\alpha\beta < 0$.

As the second example, let us consider the following nonlinear heat-transfer equation at the high excess temperature (see [14, p. 159]):

$$\frac{\partial \Xi(\xi, \tau)}{\partial \tau} = \alpha \frac{\partial^2 \Xi(\xi, \tau)}{\partial \xi^2} - \kappa \Xi^4(\xi, \tau), \quad (10)$$

where α is the thermal diffusivity, κ is a constant, and $\Xi(\xi, \tau)$ is the excess temperature.

In view of Eqs. (2), (3) and (4), we can structure the nonlinear ODE in the form:

$$-\gamma \frac{d\Xi(\omega)}{d\omega} = \alpha \frac{d^2\Xi(\omega)}{d\omega^2} - \kappa \Xi^4(\omega), \quad (11)$$

which leads to

$$\frac{d^2\Xi(\omega)}{d\omega^2} + \frac{\gamma}{\alpha} \frac{d\Xi(\omega)}{d\omega} - \frac{\kappa}{\alpha} \Xi^4(\omega) = 0. \quad (12)$$

With the aid of *MatLab software*, the solution of Eq. (12) can be written as follows:

$$\Xi(\omega) = \Lambda_1 e^{-a\omega} - \frac{b(a^5\omega^5 - 5a^4\omega^4 + 20a^3\omega^3 - 60a^2\omega^2 + 120a\omega - 120) + \Lambda_2}{5a^6} \quad (13)$$

where Λ_1 and Λ_2 are two constants, $a = \gamma / \alpha$ and $b = -\kappa / \alpha$.

Thus, clearly, we easily obtain the traveling-wave solution for Eq. (10) as follows:

$$\begin{aligned} \Xi(\xi, \tau) = & \Lambda_1 e^{-a(\xi - \gamma\tau)} - \frac{b}{5a^6} \left(a^5 (\xi - \gamma\tau)^5 - 5a^4 (\xi - \gamma\tau)^4 + 20a^3 (\xi - \gamma\tau)^3 \right) \\ & + \frac{b}{5a^6} \left(60a^2 (\xi - \gamma\tau)^2 - 120a (\xi - \gamma\tau) + 120 \right) + \frac{\Lambda_2}{5a^6}. \end{aligned} \quad (14)$$

The graphs of the traveling-wave solutions in Eq. (10) are depicted in Figs. 2(a), 2(b) and 2(c).

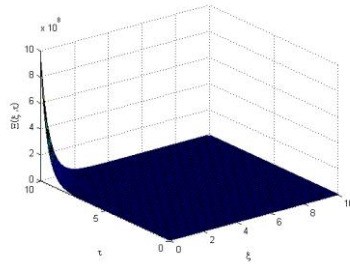


Fig.2(a). The traveling-wave solution for the nonlinear heat-transfer equation (10) for the parameters $\Lambda_1 = 1$, $a = 1$, $b = -1$, $\gamma = 1$ and $\Lambda_2 = 0$.

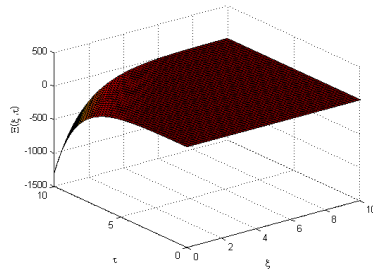


Fig. 2(b). The traveling-wave solution for the nonlinear heat-transfer equation (10) for the parameters $\Lambda_1 = 2$, $a = 1$, $b = -1$, $\gamma = 1$ and $\Lambda_2 = 0$.

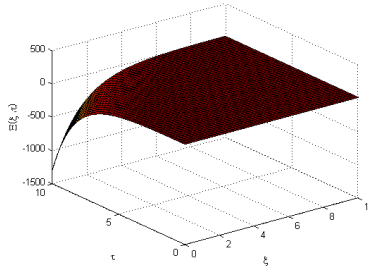


Fig.2(c). The traveling-wave solution for the nonlinear heat-transfer equation (10) for the parameters $\Lambda_1 = 1$, $a = 1$, $b = -1$, $\gamma = 2$ and $\Lambda_2 = 0$.

4. Conclusion

In our present work, we firstly investigated the linear and nonlinear heat-transfer equations at several different excess temperatures. With the help of the TTM, we transformed the linear and nonlinear PDEs arising in the heat-transfer problems into the linear and nonlinear ODEs, respectively. We then obtained the solutions of the linear and nonlinear ODEs by using the *MatLab software*. Finally, the traveling-wave solutions of these heat-transfer equations with the graphs are presented. The obtained results are given to reveal the efficiency of the techniques used in this paper.

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Nomenclature

ξ - space coordinate, [m]	α -heat-diffusion coefficient, [W/(m·K)]
$\Xi(\xi, \tau)$ -access temperature, [K]	β - a constant, [1/s]
τ - time coordinate, [s]	κ -a constant, [K ³ /s]

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