NEW EXACT SOLUTIONS OF THE (2+1)-DIMENSIONAL BROER-KAUP EQUATION BY THE CONSISTENT RICCATI EXPANSION METHOD

by

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In this work, we study the (2+1)-D Broer-Kaup equation. The composite periodic breather wave, the exact composite kink breather wave and the solitary wave solutions are obtained by using the coupled degradation technique and the consistent Riccati expansion method. These results may help us to investigate some complex dynamical behaviors and the interaction between composite non-linear waves in high dimensional models.

Key words: *(2+1)-D Broer-Kaup equation, consistent Riccati expansion method, exact solution, composite kink breather wave*

Introduction

Many non-linear phenomena in nature and human society are usually characterized by non-linear evolution equations. Searching for exact solutions of a non-linear system becomes one of the central themes with perpetual interest in non-linear science.

In this work, we consider the (2+1)-D Broer-Kaup (BK) equation in this form [1]:

$$\begin{cases} u_{ty} - u_{xxy} + 2(uu_x)_y + 2v_{xx} = 0 \\ v_t + v_{xx} + 2(uv)_x = 0 \end{cases} \tag{1}$$

Equation (1) comes from the constraints of the Kadomtsev-Petviashvili equation and it is of importance in mathematical physics field. Many researchers paid more and more attention to searching for exact solutions to (2+1)-D BK equation because of its rich physical connotations. By means of the homogeneous balance method [2], solitary wave solutions, exact multi-soliton solutions and soliton-like solutions of the BK equation were obtained. The doubly periodic wave solutions, folded solitary wave solutions, non-Lie symmetry groups and new exact solutions were derived by using the variable separation approach [3, 4], Painleve analysis method [5, 6] and the generalized reciprocal method [7], respectively.

In this paper, we will apply consistent Riccati expansion method [8] to solve the (2+1)-D BK equation.

Consistent Riccati expansion method for solving eq. (1)

Through the dependent variable transformation $v = u_y$, integrating once with respect to y and taking integration constant to zero, we obtain:

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$$u_t + u_{xx} + (u^2)_x = 0 \tag{2}$$

Equation (2) has the following truncated expansion solution [2]:

$$u = u_0 + u_1 U(X) \tag{3}$$

where u_0, u_1, and X are functions of (x, y, t) to be determined later, where $U(X)$ is a solution of the Riccati equation:

$$U_X = a_0 + a_1 U + a_2 U^2 \tag{4}$$

Therefore, the following results are obtained:

$$\begin{cases} U_1(X) = -\dfrac{a_1}{2a_2} + \dfrac{\sqrt{\delta}}{2a_2}\tanh\dfrac{\sqrt{\delta}}{2}(X + X_0), & \delta \neq 0 \\ U_2(X) = -\dfrac{a_1(X + X_0) + 2}{2a_2(X + X_0)}, & \delta = 0 \end{cases} \tag{5}$$

where a_0, a_1, and a_2 are constants, $\delta = a_1^2 - 4a_0a_2$.

Substituting eq. (4) with eq. (5) into eq. (3) and vanishing all the coefficients of $U^i (i = 0, 1, 2, 3)$, we can get:

$$u_0 = -\frac{1}{2}\left(a_1 X_x + \frac{X_t + X_{xx}}{X_x}\right), \qquad u_1 = -a_2 X_x \tag{6}$$

and

$$2X_{xx}X_{xt} + 2(X_{xx}X_t)_x - [(X_{xx} + X_t)_t + (X_t + X_{xx})_{xx}]X_x^2 -$$
$$- [X_t^2 + 3X_{xx}^2 + 4X_t X_{xx} + (4a_0a_2 - a_1^2)X_x^4 X_{xx} + (X_{xx}^2)_x + (X_t^2)_x]X_x \tag{7}$$

Suppose X is a solution of eq. (7), then:

$$u = -\frac{1}{2}\left(a_1 X_x + \frac{X_t + X_{xx}}{X_x}\right) - a_2 X_x U \tag{8}$$

Equation (8) is a solution of eq. (2) and is a solution $U = U(X)$ of the Riccati eq. (4).

To solve eq. (8), we make the following transformation:

$$X = k_1 x + k_2 y + k_3 t + f(\tau), \qquad \tau = k_4 x + k_5 y + k_6 t \tag{9}$$

where constants k_i $(i = 1, \ldots, 6)$ and function $f(\tau)$ are unknown to be determined.

Substituting eq. (9) into eq. (8), we get:

$$\delta k_4^5(k_4 f' + 4k_1)f''f'^3 + k_4^4(6\delta k_1^2 f'' - k_4^2 f^{(4)})f'^2 +$$
$$+ 2k_4^3[2(k_4^3 f^{(3)} + \delta k_1^3)f'' + \mu f^{(3)} - k_1 k_4^2 f^{(4)}]f' - k_4^3(3k_4^3 f'' + 4\mu)f''^2 +$$
$$+ [k_1 k_4^2(k_1^3\delta + 4k_4^3)f^{(3)} - \mu^2]f'' + k_1 k_4^2(2\mu f - k_1 k_4^2 f')^{(3)} = 0 \tag{10}$$

where $\mu = k_3k_4 - k_1k_6$, and $f' = [df(\tau)]/d\tau$. Let $f' = g(\tau)$, eq. (10) can be written:

$$\delta k_4^5(k_4 g+4k_1)g'g^3+k_4^4(6\delta k_1^2 g'-k_4^2 g''')g^2+$$
$$+2k_4^3[2(k_4^3 g''+\delta k_1^3)g'+\mu g''-k_1k_4^2 g''']g-k_4^3(3k_4^3 g'+4\mu)g'^2+$$
$$+[k_1k_4^2(k_1^3\delta+4k_4^3)g''-\mu^2]g'+k_1k_4^2(2\mu g-k_1k_4^2 g')''=0 \tag{11}$$

We derive the solution of eq. (11) in the following forms:

$$g(\tau)=b_0+b_1 h(\tau) \tag{12}$$

where $h(\tau)$ satisfies the following Riccati equation:

$$h'=c_0+c_1 h+c_2 h^2 \tag{13}$$

Equation (12) has the following solutions:

$$\begin{cases} h_1(\tau)=-\dfrac{c_1}{2c_2}+\dfrac{\sqrt{\rho}}{2a_2}\tanh\dfrac{\sqrt{\rho}}{2}(\tau+\tau_0), & \rho\neq 0 \\ h_2(\tau)=-\dfrac{c_1(\tau+\tau_0)+2}{2c_2(\tau+\tau_0)}, & \rho=0 \end{cases} \tag{14}$$

where $\rho = c_1^2 - 4c_0c_2$, b_0, b_1 are constants to be determined.

Substituting eq. (13) with eq. (14) into eq. (12) and setting the coefficients of $h^j (j = 0, 1, \dots, 6)$ to zero, we can get a set of non-linear algebraic equations. Solving these equations by MAPLE, we obtain the solutions:

$$b_0=-\frac{c_1k_4+2k_1\sqrt{\delta}}{2k_4\sqrt{\delta}},\qquad b_1=\frac{c_1^2}{8c_0\sqrt{\delta}},\qquad k_3=\frac{8k_1k_6\sqrt{\delta}-3c_1^2k_4^3}{8k_4\sqrt{\delta}},\qquad c_2=-\frac{c_1^2}{8c_0} \tag{15}$$

where $k_4k_5k_6c_0c_1\delta \neq 0$ and $k_1, k_2 \in R$.

Substituting eq. (14) with eq. (15) into eq. (12), we get the solution of eq. (12):

$$\begin{cases} g_1(\tau)=\alpha+\dfrac{c_1^2\sqrt{\rho}}{16c_0a_2\sqrt{\delta}}\tanh\dfrac{\sqrt{\rho}}{2}(\tau+\tau_0), & \rho\neq 0 \\ g_2(\tau)=-\dfrac{c_1k_4+2k_1\sqrt{\delta}}{2k_4\sqrt{\delta}}-\dfrac{c_1^2[c_1(\tau+\tau_0)+2]}{16c_0c_2\sqrt{\delta}(\tau+\tau_0)}, & \rho=0 \end{cases} \tag{16}$$

where

$$\alpha=-\frac{c_1k_4+2k_1\sqrt{\delta}}{2k_4\sqrt{\delta}}-\frac{c_1^3}{16c_0c_2\sqrt{\delta}}$$

whose corresponding $f(\tau)$ is given by:

$$\begin{cases} f_1(\tau)=\alpha\tau-\dfrac{c_1^2}{16c_0a_2\sqrt{\delta}}\ln\left[\tanh^2\dfrac{\sqrt{\rho}}{2}(\tau+\tau_0)-1\right], & \rho\neq 0 \\ f_2(\tau)=-\left[\dfrac{c_1k_4+2k_1\sqrt{\delta}}{2k_4\sqrt{\delta}}+\dfrac{c_1^3}{16c_0c_2\sqrt{\delta}}\right]\tau-\dfrac{c_1^2\ln(\tau+\tau_0)}{8c_0c_2\sqrt{\delta}}, & \rho=0 \end{cases} \tag{17}$$

Substituting eq. (16) with eq. (17) into eq. (11), we have the solution of eq. (8):

$$\begin{cases} X_1 = \alpha_1 x + \alpha_2 y + \alpha_3 t - \sigma \ln\left[\tanh^2 \dfrac{\sqrt{\rho}}{2}(k_4 x + k_5 y + k_6 t + \tau_0) - 1\right], & \rho \neq 0 \\ X_2 = \beta_1 x + \beta_2 y + \beta_3 t - \dfrac{c_1^2 \ln(k_4 x + k_5 y + k_6 t + \tau_0)}{8 c_0 c_2 \sqrt{\delta}}, & \rho = 0 \end{cases} \tag{18}$$

where

$$\alpha_i = k_i + \alpha k_{i+3}, \quad \beta_i = k_i - \left(\frac{c_1 k_4 + 2k_1\sqrt{\delta}}{2k_4\sqrt{\delta}} + \frac{c_1^3}{16 c_0 c_2 \sqrt{\delta}}\right) k_{i+3}, \quad i = 1,2,3, \quad \sigma = \frac{c_1^2}{16 c_0 a_2 \sqrt{\delta}}$$

Substituting eq. (18) with eq. (5) into eq. (8), we have solutions of eq. (2):

$$u_{11} = \frac{2\alpha_3 + \sigma[k_4^2 \rho(1 - \tanh^2 \xi) + 2k_6\sqrt{\rho}\tanh\xi]}{4(\alpha_1 + k_4\sigma\sqrt{\rho}\tanh\xi)} - \\ -\frac{\sqrt{\delta}}{2}(\alpha_1 + k_4\sigma\sqrt{\rho}\tanh\xi)\tanh\frac{\sqrt{\delta}}{2}[\eta - \sigma\ln(\tanh^2\xi - 1) + X_0)] \tag{19}$$

$$u_{12} = \frac{32 c_0 c_2 \beta_3 \sqrt{\delta}\sqrt{\rho}\xi^2 - 2c_1^2 k_6 \xi + c_1^2 k_4^2 \sqrt{\rho}}{4\sqrt{\rho}\xi(16 c_0 c_2 \beta_1 \xi - c_1^2 k_4)} - \\ -\frac{\sqrt{\delta}}{2}\left(\beta_1 - \frac{c_1^2 k_4 \sqrt{\rho}}{16 c_0 c_2 \sqrt{\delta}\xi}\right)\tanh\frac{\sqrt{\delta}}{2}\left(\eta - \frac{c_1^2}{8 c_0 c_2 \sqrt{\delta}}\ln\frac{2}{\sqrt{\rho}}\xi + X_0\right) \tag{20}$$

$$u_{21} = -\frac{2\alpha_3 + \sigma[k_4^2\rho(1 - \tanh^2\xi) + 2k_6\sqrt{\rho}\tanh\xi]}{4[\alpha_1 + k_4\sigma\sqrt{\rho}\tanh\xi]} + \frac{\alpha_1 + k_4\sigma\sqrt{\rho}\tanh\xi}{\eta - \sigma\ln(\tanh^2\xi - 1) + X_0} \tag{21}$$

$$u_{22} = -\frac{32 c_0 c_2 \beta_3 \sqrt{\delta}\sqrt{\rho}\xi^2 - 2c_1^2 k_6\xi + c_1^2 k_4^2\sqrt{\rho}}{4\sqrt{\rho}\xi[16 c_0 c_2 \beta_1 \xi - c_1^2 k_4]} + \\ + \frac{16 c_0 c_2 \beta_1 \sqrt{\delta}\xi - c_1^2 k_4 \sqrt{\rho}}{2\xi\left[8 c_0 c_2 \sqrt{\delta}\eta - c_1^2 \ln\dfrac{2}{\sqrt{\rho}}\xi + 8 c_0 c_2 \sqrt{\delta} X_0\right]} \tag{22}$$

Through the transformation $v = u_v$, we can obtain v_{11}, v_{12}, v_{21}, v_{22}, respectively:

$$v_{11} = \left\{\begin{matrix} \dfrac{2\alpha_3 + \sigma[k_4^2\rho(1 - \tanh^2\xi) + 2k_6\sqrt{\rho}\tanh\xi]}{4(\alpha_1 + k_4\sigma\sqrt{\rho}\tanh\xi)} - \\ -\dfrac{\sqrt{\delta}}{2}(\alpha_1 + k_4\sigma\sqrt{\rho}\tanh\xi)\tanh\dfrac{\sqrt{\delta}}{2}[\eta - \sigma\ln(\tanh^2\xi - 1) + X_0)] \end{matrix}\right\}_y \tag{23}$$

$$v_{12}=\left\{\begin{array}{l}\dfrac{32c_0c_2\beta_3\sqrt{\delta}\sqrt{\rho}\xi^2-2c_1^2k_6\xi+c_1^2k_4^2\sqrt{\rho}}{4\sqrt{\rho}\xi(16c_0c_2\beta_1\xi-c_1^2k_4)}-\\ -\dfrac{\sqrt{\delta}}{2}\left(\beta_1-\dfrac{c_1^2k_4\sqrt{\rho}}{16c_0c_2\sqrt{\delta}\xi}\right)\tanh\dfrac{\sqrt{\delta}}{2}\left(\eta-\dfrac{c_1^2}{8c_0c_2\sqrt{\delta}}\ln\dfrac{2}{\sqrt{\rho}}\xi+X_0\right)\end{array}\right\}_y \tag{24}$$

$$v_{21}=\left\{-\frac{2\alpha_3+\sigma[k_4^2\rho(1-\tanh^2\xi)+2k_6\sqrt{\rho}\tanh\xi]}{4(\alpha_1+k_4\sigma\sqrt{\rho}\tanh\xi)}+\frac{\alpha_1+k_4\sigma\sqrt{\rho}\tanh\xi}{\eta-\sigma\ln(\tanh^2\xi-1)+X_0}\right\}_y \tag{25}$$

$$v_{22}=\left\{-\frac{32c_0c_2\beta_3\sqrt{\delta}\sqrt{\rho}\xi^2-2c_1^2k_6\xi+c_1^2k_4^2\sqrt{\rho}}{4\sqrt{\rho}\xi(16c_0c_2\beta_1\xi-c_1^2k_4)}+\frac{16c_0c_2\beta_1\sqrt{\delta}\xi-c_1^2k_4\sqrt{\rho}}{2\xi\left[8c_0c_2\sqrt{\delta}\eta-c_1^2\ln\dfrac{2}{\sqrt{\rho}}\xi+8c_0c_2\sqrt{\delta}X_0\right]}\right\}_y \tag{26}$$

where $\eta=\alpha_1x+\alpha_2y+\alpha_3t,\ \xi=(\rho^{1/2}/2)(k_4x+k_5y+k_6t+\tau_0)$.

Obvious, the $\delta>0$ in:

$$U_1(X)=-\frac{a_1}{2a_2}+\frac{\sqrt{\delta}}{2a_2}\tanh\frac{\sqrt{\delta}}{2}(X+X_0)$$

of eq. (5). If the $\delta<0$, then:

$$U_1(X)=-\frac{a_1}{2a_2}+\frac{\sqrt{\delta}}{2a_2}\tan\frac{\sqrt{\delta}}{2}(X+X_0)$$

in which is the period wave solution.

Similarly, the $\rho>0$ in:

$$h_1(\tau)=-\frac{c_1}{2c_2}+\frac{\sqrt{\rho}}{2a_2}\tanh\frac{\sqrt{\rho}}{2}(\tau+\tau_0)$$

of eq. (14). If the $\rho<0$, then:

$$h_1(\tau)=-\frac{c_1}{2c_2}+\frac{\sqrt{\rho}}{2a_2}\tan(\tau+\tau_0)$$

in which is the period wave solution too.

Accordingly, previous composite solution of kink wave and kink wave (19)-(26) are conversion to the composite solution of period wave and period wave, the composite solution of period wave and kink wave and the composite solution of kink wave and period wave, respectively.

Conclusion

In this paper, the coupled (2+1)-D BK equation with consistent Riccati expansion method has been successfully implemented to obtain the new composite exact solutions structures, including the composite periodic breather wave, the kink breather wave, and the solitary wave solutions. The results we have achieved here may help us to investigate the local structure and interaction of the composite non-linear waves in this model. Further study is needed to discover more dynamic features of the (2+1)-D BK equation. Moreover, the consistent Riccati expansion method has been proved to be a powerful, efficient technique for finding the exact solutions of the coupled degradation systems.

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