

## AN ANALYTIC STUDY ON THE TWO-TEMPERATURE MODEL FOR ELECTRON-LATTICE THERMAL DYNAMIC PROCESS

by

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*In this paper, we study the TTM arising in electron-lattice thermal dynamic process by two methods. A new exact traveling solution and variable separation solutions are obtained. They can help us to understand morphological differences in femtosecond laser inducing periodic surface structures on noble metals. Our study examines the role of two competing ultrafast processes following femtosecond laser heating of metals thoroughly. The calculation results confirm the previous experimental work, which is the electron-phonon coupling strength plays a dominant role in the process.*

Key words: TTM, noble metals, traveling wave solutions, variable separation solutions

### Introduction

Ultrashort pulse laser has found increasing applications in broad fields including material processing, chemistry, biology, health, medicine, electronic, and information technology. The ultrashort pulse duration, usually in the order of femtoseconds to hundreds of femtoseconds, presents its unique properties such as very high peak power, very limited thermal effects, compared with long pulse laser. The interaction mechanism between ultrashort pulse laser and materials is of importance due to its more and more broad applications. In the theories describing material changes after ultrashort laser pulse radiation, the two-temperature model (TTM) [1] is well-known and widely used. In this model, the absorption is divided into two processes. When ultrashort laser pulses arrive at the surface of a metal, the free electrons in the metal first absorb the energy of the laser light, that is, photons. Then there are two paths for the energy absorbed by the electrons to rearrange the thermal diffusion of the hot electrons and the collisions between the electrons and phonons. The latter interaction makes the material temperature rise. Based on this consideration, Anisimov *et al.* [2] obtained the TTM. This electron-lattice thermal dynamic process described by the TTM is:

$$\begin{cases} C_e \frac{\partial T_e}{\partial t} = \nabla(K_e \nabla T_e) - g(T_e - T_l) + S(r, t) \\ C_l \frac{\partial T_l}{\partial t} = \nabla(K_l \nabla T_l) + g(T_e - T_l) \end{cases} \quad (1)$$

where  $T_e, T_l: R_r \times R_z \times R_t \rightarrow R$ ,  $C_e$ , and  $C_l$  are the electron and lattice heat capacity,  $K_e$ , and  $K_l$  – the electron and lattice thermal conductivity,  $g$  – the electron-phonon coupling coefficient

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cient,  $T_e$ , and  $T_l$  – the electron and lattice temperatures, respectively, and  $S$  is the initial energy deposition function by the external source. The spatial and time evolution of electron and lattice temperature can then be obtained by solving the coupled differential equations in eq. (1). This equation set is usually solved by numerical methods.

In this paper, we use some appropriate methods [3-10] to study on the exact solutions of the TTM (1).

### Traveling wave solutions of TTM as $S(r, t) = 0$

In this section, we introduce independent variables at first:

$$T_e(r, z, t) = u(\xi), \quad T_l(r, z, t) = v(\xi), \quad \xi = kr + lz + ct \quad (2)$$

Equation (1) is carried to ordinary differential equations (ODE) of  $\xi$ :

$$\begin{cases} C_e cu' - K_e su'' + g(u - v) = 0 \\ C_l cv' - K_l sv'' - g(u - v) = 0 \end{cases} \quad (3)$$

where  $u' = du(\xi)/d\xi$ ,  $v' = dv(\xi)/d\xi$ ,  $s = k^2 + l^2 + c^2 \neq 0$ , and  $k, l, c$  are non-zero real constants to be determined.

*Theorem 1.* If  $S(r, t) = 0$  and  $\lambda_1, \lambda_2$ , and  $\lambda_3$  are roots of this algebraic equation:

$$K_e K_l s^2 \lambda^3 - cs(C_e K_l + C_l K_e) \lambda^2 + [C_e C_l c^2 - gs(K_e - K_l)] \lambda + gc(C_e + C_l) = 0 \quad (4)$$

then the general solution of eq. (1) is:

$$\begin{cases} T_e(r, z, t) = c_0 + \mu_1 e^{\lambda_1(kr+lz+ct)} + \mu_2 e^{\lambda_2(kr+lz+ct)} + \mu_3 e^{\lambda_3(kr+lz+ct)} \\ T_l(r, z, t) = c_0 + c_1 e^{\lambda_1(kr+lz+ct)} + c_2 e^{\lambda_2(kr+lz+ct)} + c_3 e^{\lambda_3(kr+lz+ct)} \end{cases} \quad (5)$$

where  $\mu_i = c_i[1 + \lambda_i(C_l c - K_l s \lambda_i)/g]$ , and  $c_i (i=1, 2, 3)$  are arbitrary real constants.

*Proof.* First, we solve the second equation in eq. (3). We get:

$$u = \frac{C_l cv' - K_l sv''}{g + v} \quad (6)$$

Substituting eq. (6) into the first equation in eq. (3), we get a linear four order ordinary differential equation:

$$K_e K_l s^2 v'''' - cs(C_e K_l + C_l K_e) v'''' + [C_e C_l c^2 - gs(K_e - K_l)] v'' + gc(C_e + C_l) v' = 0 \quad (7)$$

Assume that the characteristic roots are  $\lambda$ , then the characteristic equation is:

$$K_e K_l s^2 \lambda^4 - cs(C_e K_l + C_l K_e) \lambda^3 + [C_e C_l c^2 - gs(K_e - K_l)] \lambda^2 + gc(C_e + C_l) \lambda = 0 \quad (8)$$

Obviously,  $\lambda = 0$  is a root of eq. (7). If the non-zero roots are  $\lambda_1, \lambda_2$ , and  $\lambda_3$ , one can get the solution of ODE (7):

$$v(\xi) = c_0 + c_1 e^{\lambda_1 \xi} + c_2 e^{\lambda_2 \xi} + c_3 e^{\lambda_3 \xi} \quad (9)$$

Substituting eq. (9) into eq. (2), we have:

$$T_1(r, z, t) = c_0 + c_1 e^{\lambda_1(kr+lz+ct)} + c_2 e^{\lambda_2(kr+lz+ct)} + c_3 e^{\lambda_3(kr+lz+ct)} \quad (10)$$

where  $c_i (i=1,2,3)$  are arbitrary real constants. Substituting eq. (10) into eq. (6) with eq. (2), yields:

$$T_e(r, z, t) = c_0 + \mu_1 e^{\lambda_1(kr+lz+ct)} + \mu_2 e^{\lambda_2(kr+lz+ct)} + \mu_3 e^{\lambda_3(kr+lz+ct)} \quad (11)$$

where  $\mu_i = c_i[1 + \lambda_i(C_1 c - K_1 s \lambda_i)/g]$  ( $i=1, 2, 3$ ), in this way we have completed the proof of *Theorem 1*.

Specially, let  $c_3 = 0$  and  $\lambda_2 = -\lambda_1$  in eq. (10), then:

*Case 1.* If  $c_1 c_2 > 0$  and  $\lambda_1 > 0$ , we have:

$$v(\xi) = c_0 + 2\sqrt{c_1 c_2} \cosh\left(\lambda_1 \xi + \frac{1}{2} \ln \frac{c_2}{c_1}\right) \quad (12)$$

Similarly yields:

$$T_1(r, z, t) = c_0 + 2\sqrt{c_1 c_2} \cosh\left[\lambda_1(kr+lz+ct) + \frac{1}{2} \ln \frac{c_2}{c_1}\right] \quad (13)$$

Substituting eq. (10) into eq. (6) with eq. (2), yields:

$$T_e(r, z, t) = c_0 + \mu_1 \sinh\left[\lambda_1(kr+lz+ct) + \frac{1}{2} \ln \frac{c_2}{c_1}\right] + \mu_2 \cosh\left[\lambda_1(kr+lz+ct) + \frac{1}{2} \ln \frac{c_2}{c_1}\right] \quad (14)$$

where  $\mu_1 = \frac{2C_1 c \lambda_1}{g} \sqrt{c_1 c_2}$  and  $\mu_2 = 2\left(1 - \frac{K_1 s \lambda_1^2}{g}\right) \sqrt{c_1 c_2}$ .

*Case 2.* If  $c_1 c_2 < 0$  and  $\lambda_1 > 0$ , then we have:

$$v(\xi) = c_0 - 2\sqrt{-c_1 c_2} \sinh\left(\lambda_1 \xi + \frac{1}{2} \ln \left|\frac{c_2}{c_1}\right|\right) \quad (15)$$

Similarly we have:

$$T_1(r, z, t) = c_0 - 2\sqrt{-c_1 c_2} \sinh\left[\lambda_1(kr+lz+ct) + \frac{1}{2} \ln \left|\frac{c_2}{c_1}\right|\right] \quad (16)$$

Substituting eq. (16) into eq. (6) with eq. (2), yields:

$$T_e(r, z, t) = c_0 + \mu_1 \cosh\left[\lambda_1(kr+lz+ct) + \frac{1}{2} \ln \left|\frac{c_2}{c_1}\right|\right] + \mu_2 \sinh\left[\lambda_1(kr+lz+ct) + \frac{1}{2} \ln \left|\frac{c_2}{c_1}\right|\right] \quad (17)$$

where  $\mu_1 = -\frac{2C_1 c \lambda_1}{g} \sqrt{-c_1 c_2}$  and  $\mu_2 = 2\left(\frac{K_1 s \lambda_1^2}{g} - 1\right) \sqrt{-c_1 c_2}$ .

### Variable separation solution of TTM as $S(r, t) \neq 0$

*Theorem 2.* If  $C_e = \lambda K_e$ ,  $C_1 = \lambda K_1$ ,  $K_e \neq K_1$  and:

$$S(r, t) = (c_3 e^{\rho t} + c_4 e^{\tau t}) \left( c_5 \sin \sqrt{\frac{g}{K_e}} r + c_6 \cos \sqrt{\frac{g}{K_e}} r \right) \quad (18)$$

then the variable separation solution of eq. (1) is:

$$\begin{cases} T_e(r, z, t) = \left[ c_1 \sin \kappa z + c_2 \cos \kappa z - \frac{L_1}{2g(K_e - K_1)} \right] (c_3 e^{\rho t} + c_4 e^{\tau t}) \left( c_5 \sin \sqrt{\frac{g}{K_e}} r + c_6 \cos \sqrt{\frac{g}{K_e}} r \right) \\ T_1(r, z, t) = \left[ c_1 \sin \kappa z + c_2 \cos \kappa z - \frac{L_e}{2g(K_e - K_1)} \right] (c_3 e^{\rho t} + c_4 e^{\tau t}) \left( c_5 \sin \sqrt{\frac{g}{K_e}} r + c_6 \cos \sqrt{\frac{g}{K_e}} r \right) \end{cases} \quad (19)$$

where

$$\kappa = \sqrt{\frac{g(K_e - K_1)}{K_e K_1}}, \quad \rho = \frac{K_1 \lambda + \sqrt{K_1^2 \lambda^2 + 4gK_1}}{2k_1}, \quad \tau = \frac{K_1 \lambda - \sqrt{K_1^2 \lambda^2 + 4gK_1}}{2k_1}, \quad c_i (i=1 \dots 6)$$

are arbitrary real constants and  $(c_1^2 + c_2^2)(c_3^2 + c_4^2)(c_5^2 + c_6^2) \neq 0$ .

*Proof.* Let  $f = f(z)$  and suppose:

$$T_e(r, z, t) = \left( f + \frac{1}{2g} \right) S, \quad T_1(r, z, t) = f S \quad (20)$$

Substituting eq. (20) into eq. (1), one can get:

$$\begin{cases} C_e \left( f + \frac{1}{2g} \right) S_t - K_e \left[ \left( f + \frac{1}{2g} \right) (S_{rr} + S_{tt}) + S f_{zz} \right] - \frac{1}{2} S = 0 \\ C_1 f S_t - K_1 [f (S_{rr} + S_{tt}) + S f_{zz}] - \frac{1}{2} S = 0 \end{cases} \quad (21)$$

Taking  $C_e = \lambda K_e$ ,  $C_1 = \lambda K_1$  in eq. (21), yields:

$$\begin{cases} K_e \left( f + \frac{1}{2g} \right) (\lambda S_t - S_{rr} - S_{tt}) = \frac{1}{2} S (1 + 2K_e f_{zz}) \\ K_1 f (\lambda S_t - S_{rr} - S_{tt}) = \frac{1}{2} S (1 + 2K_1 f_{zz}) \end{cases} \quad (22)$$

$$\text{Namely} \quad \frac{\lambda S_t - S_{rr} - S_{tt}}{S} = \frac{g(1 + 2K_e f_{zz})}{K_e(2gf + 1)}, \quad \frac{\lambda S_t - S_{rr} - S_{tt}}{S} = \frac{1 + 2K_1 f_{zz}}{2K_1} \quad (23)$$

$$\text{Thus} \quad \frac{g(1 + 2K_e f_{zz})}{K_e(2gf + 1)} = \frac{1 + 2K_1 f_{zz}}{2K_1},$$

$$\text{we have} \quad 2K_e K_1 f_{zz} + 2g(K_e - K_1)f + K_e = 0 \quad (24)$$

Solving the ODE (24), we get:

$$f(z) = c_1 \sin \kappa z + c_2 \cos \kappa z - \frac{K_e}{2g(K_e - K_1)} \quad (25)$$

where  $\kappa = g(K_e - K_1)/(K_e K_1)$ .

Substituting eq. (25) into eq. (23) yields linear partial differential equation:

$$K_e K_1 (\lambda S_t - S_{rr} - S_{tt}) + g(K_e - K_1) S = 0 \quad (26)$$

We assume that:

$$S(r, t) = m(t)h(r) \quad (27)$$

where  $m(t)$  is a function of  $t$ , and  $h(r)$  is a function of  $r$  that have to be determined.

Substituting eq. (27) into eq. (26), we have:

$$K_e h(gm + \lambda K_1 m' - K_1 m'') - K_1 m(gh + K_e h'') = 0 \quad (28)$$

Letting  $gm + \lambda K_1 m' - K_1 m'' = 0$  and  $gh + K_e h'' = 0$  in eq. (28), we obtain:

$$m(t) = c_3 e^{\rho t} + c_4 e^{\tau t}, \quad h(r) = c_5 \sin \sqrt{\frac{g}{K_e}} r + c_6 \cos \sqrt{\frac{g}{K_e}} r \quad (29)$$

Then 
$$S(r, t) = (c_3 e^{\rho t} + c_4 e^{\tau t}) \left( c_5 \sin \sqrt{\frac{g}{K_e}} r + c_6 \cos \sqrt{\frac{g}{K_e}} r \right) \quad (30)$$

where 
$$\rho = \frac{K_1 \lambda + \sqrt{K_1^2 \lambda^2 + 4gK_1}}{2k_1}, \quad \tau = \frac{K_1 \lambda - \sqrt{K_1^2 \lambda^2 + 4gK_1}}{2k_1}.$$

Substituting eqs. (30) and (25) into eq. (20), we obtain eq. (19). In this way we have completed the proof of *Theorem 2*.

If for  $g(K_e - K_1)K_e K_1 < 0$ , we set:

$$\kappa = \sqrt{\frac{g(K_e - K_1)}{K_e K_1}} = i\kappa_0, \quad c_1 = -ic_{10}, \quad (c_{10}, \kappa_0 \in \mathbb{R} - \{0\}, i^2 = -1)$$

then the solutions (19) can be converted in the following form:

$$\begin{cases} T_e(r, z, t) = \left( c_{10} \sinh \kappa_0 z + c_2 \cosh \kappa_0 z - \frac{L_1}{2g(K_e - K_1)} \right) (c_3 e^{\rho t} + c_4 e^{\tau t}) \left( c_5 \sin \sqrt{\frac{g}{K_e}} r + c_6 \cos \sqrt{\frac{g}{K_e}} r \right) \\ T_1(r, z, t) = \left( c_{10} \sinh \kappa_0 z + c_2 \cosh \kappa_0 z - \frac{L_e}{2g(K_e - K_1)} \right) (c_3 e^{\rho t} + c_4 e^{\tau t}) \left( c_5 \sin \sqrt{\frac{g}{K_e}} r + c_6 \cos \sqrt{\frac{g}{K_e}} r \right) \end{cases} \quad (31)$$

Also, if for  $gK_g < 0$ , we set  $\sqrt{\frac{g}{K_e}} = \theta i$ ,  $c_5 = -ic_{50}$ ,  $(c_{50}, \theta \in \mathbb{R} - \{0\}, i^2 = -1)$

solutions (19) can be converted in the following form:

$$\begin{cases} T_e(r, z, t) = \left( c_1 \sin \kappa z + c_2 \cos \kappa z - \frac{L_1}{2g(K_e - K_1)} \right) (c_3 e^{\rho t} + c_4 e^{\tau t}) (c_{50} \sinh \theta r + c_6 \cosh \theta r) \\ T_1(r, z, t) = \left( c_1 \sin \kappa z + c_2 \cos \kappa z - \frac{L_e}{2g(K_e - K_1)} \right) (c_3 e^{\rho t} + c_4 e^{\tau t}) (c_{50} \sinh \theta r + c_6 \cosh \theta r) \end{cases} \quad (32)$$

From eq. (31) and eq. (32), one can get:

$$\begin{cases} T_c(r, z, t) = \left( c_{10} \sinh \kappa_0 z + c_2 \cosh \kappa_0 z - \frac{L_1}{2g(K_e - K_1)} \right) (c_3 e^{\rho t} + c_4 e^{\tau t}) (c_{50} \sinh \theta r + c_6 \cosh \theta r) \\ T_1(r, z, t) = \left( c_{10} \sinh \kappa_0 z + c_2 \cosh \kappa_0 z - \frac{L_e}{2g(K_e - K_1)} \right) (c_3 e^{\rho t} + c_4 e^{\tau t}) (c_{50} \sinh \theta r + c_6 \cosh \theta r) \end{cases} \quad (33)$$

The solutions obtained in this paper are new.

### Conclusion

In this paper, we used two methods to study the TTM, and obtained a new exact traveling solution and variable separation solutions for this system. Results of eqs. (4) and (11) include the abundant solution structures and physical properties of model (1). Our study thoroughly examined the role of two competing ultrafast processes following femtosecond laser heating of metals. The calculation results confirmed our previous experimental work on femtosecond laser inducing periodic surface structures on noble metals, where the electron-phonon coupling strength was believed to play the dominant role. The profound calorific significance of some parameters in results of eqs. (4) and (11) needs further study.

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