LUMP SOLUTIONS TO THE (2+1)-DIMENSIONAL SHALLOW WATER WAVE EQUATION

by

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Through symbolic computation with MAPLE, a class of lump solutions to the (2+1)-D shallow water wave equation is presented, making use of its Hirota bilinear form. The resulting lump solutions contain six free parameters, two of which are due to the translation invariance of the (2+1)-D shallow water wave equation and the other four of which satisfy a non-zero determinant condition guaranteeing analyticity and rational localization of the solutions.

Key words: *lump solution, Hirota bilinear, shallow water wave equation*

Introduction

In recent years, there has been a growing interest in rational solutions to non-linear differential equations [1, 2], which are used to describe the mechanical, process control, ecological, and economic systems, chemical recycling system, and other areas of epidemiology issues.

The Hirota bilinear method [3] makes use of the bilinear derivative, which only concerned with solving equations, rather than relying on spectrum issues or Lax pairs [4]. Hirota bilinear forms play an important role in presenting soliton solutions [5, 6], though some intelligent guesswork is often necessary [7].

In this paper, we would like to focus on the (2+1)-D shallow water wave equation has a Hirota bilinear form [8, 9] and presents a general class of lump solutions by symbolic computation with MAPLE.

A shallow water wave-like equation

Let us consider the (2+1)-D shallow water wave equation [10]:

$$u_{xxy} + \frac{2}{3}u_x u_y - u_y - u_t = 0 \tag{1}$$

Equation (1) becomes the following Hirota bilinear equation under transformation $u = 2(\ln f)_x$:

$$\begin{aligned}&(D_x^3 D_y - D_x D_y - D_x D_t)ff = \\ &= 2f_{xxxy}f + 6f_{xx}f_{xy} - 6f_x f_{xxy} - 2f_{xxx}f_y - 2f_{xy}f - 2f_x f_y + 2f_{xt}f - 2f_x f_t = \\ &= 0\end{aligned} \tag{2}$$

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where f solves eq. (2) if and only if $u = 2(\ln f)_x$ presents a solution to the (2+1)-D shallow water wave eq. (1). To search for quadratic function solutions to the (2+1)-D shallow water wave equation in eq. (2), we begin with:

$$f = g^2 + h^2 + a_9, \qquad g = a_1 x + a_2 y + a_3 t + a_4, \qquad h = a_5 x + a_6 y + a_7 t + a_8 \tag{3}$$

where a_1, (i = 1….9), are real parameters to be determined [11, 12]. A simple form $g^2 + a_5$ does not generate analytic solutions, which are rationally localized in all directions in the space, and so, we start with eq. (3). A direct MAPLE symbolic computation with f above generates the following set of constraining equations for the parameters:

$$\begin{gathered} a_1 = a_1, \quad a_2 = a_2, \quad a_3 = \frac{1}{a_1^2 + a_5^2}, \quad a_4 = a_4, \quad a_5 = a_5, \\ a_6 = a_6, \quad a_7 = \frac{1}{a_1^2 + a_5^2}, \quad a_8 = a_8, \quad a_9 = \frac{1}{(a_1 a_6 - a_2 a_5)^2} \end{gathered} \tag{4}$$

which needs to satisfy a determinant condition:

$$\Delta = a_1 a_6 - a_2 a_5 = \begin{vmatrix} a_1 & a_2 \\ a_5 & a_6 \end{vmatrix} \neq 0 \tag{5}$$

This set leads to a class of positive quadratic function solutions to eq. (2):

$$f = \left(a_1 x + a_2 y + \frac{1}{a_1^2 + a_5^2} t + a_4 \right)^2 + \left(a_5 x + a_6 y + \frac{1}{a_1^2 + a_5^2} t + a_8 \right)^2 + \frac{1}{(a_1 a_6 - a_2 a_5)^2} \tag{6}$$

and the resulting class of quadratic function solutions, in turn, yields a class of lump solutions to the (2+1)-D shallow water wave equation in eq. (1) through the transformation:

$$u = \frac{2a_1 g + 2a_5 h}{f} \tag{7}$$

where the function f is defined by eq. (6) and the function of g and h are given:

$$g = a_1 x + a_2 y + \frac{1}{a_1^2 + a_5^2} t + a_4 \tag{8}$$

$$h = a_5 x + a_6 y + \frac{1}{a_1^2 + a_5^2} t + a_8 \tag{9}$$

In this class of lump solutions, all six involved parameters of a_1, a_2, a_4, a_5, a_6, and a_8 are arbitrary provided that the solutions are well defined [13], *i. e.*, if the determinant condition eq. (5) is satisfied. That determinant condition precisely means that two directions (a_1, a_2) and (a_5, a_6) in the xy – plane are not parallel [14].

When the condition of eq. (5) is satisfied, we can get this information that $a_1^2 + a_5^2 \neq 0$, so $a_9 > 0$. This leads to the analyticity of solutions in eq. (6) which is analytic in the x-y plane if and only if the parameter $a_9 > 0$. At any time, t, lump solutions $u \to 0$ if and only if the corresponding sum of squares $g^2 + h^2 \to \infty$ or, in equivalent $x^2 + y^2 \to \infty$ [15, 16]. Therefore, both analyticity and localization of the solutions in eq. (7) is guaranteed when the condition of eq. (5) is satisfied.

If we take two special choices for the parameters, we get the following lump solutions. First, a selection of the parameters:

$$a_1 = -1, \qquad a_2 = -\frac{1}{2}, \qquad a_4 = 0, \qquad a_5 = 1, \qquad a_6 = -\frac{1}{2}, \qquad a_8 = 0 \tag{10}$$

leads to

$$f = \left(-x - \frac{1}{2}y + \frac{1}{2}t\right)^2 + \left(x - \frac{1}{2}y + \frac{1}{2}t\right)^2 + 1 \tag{11}$$

$$u = \frac{2(4x-2)}{\left(-x - \frac{1}{2}y + \frac{1}{2}t\right)^2 + \left(x - \frac{1}{2}y + \frac{1}{2}t\right)^2 + 1} \tag{12}$$

Second, another selection of the parameters:

$$a_1 = 2, \qquad a_2 = -\frac{1}{8}, \qquad a_4 = 0, \qquad a_5 = -2, \qquad a_6 = -\frac{1}{8}, \qquad a_8 = 0 \tag{13}$$

Yields:

$$f = \left(2x - \frac{1}{8}y + \frac{1}{8}t\right)^2 + \left(-2x - \frac{1}{8}y + \frac{1}{8}t\right)^2 + 4 \tag{14}$$

$$u = \frac{32x}{\left(2x - \frac{1}{8}y + \frac{1}{8}t\right)^2 + \left(-2x - \frac{1}{8}y + \frac{1}{8}t\right)^2 + 4} \tag{15}$$

When $t = 1$, the plots of eq. (12) are depicted in figs. 1 and 2, the plots of eq. (15) are depicted in figs. 3 and 4.

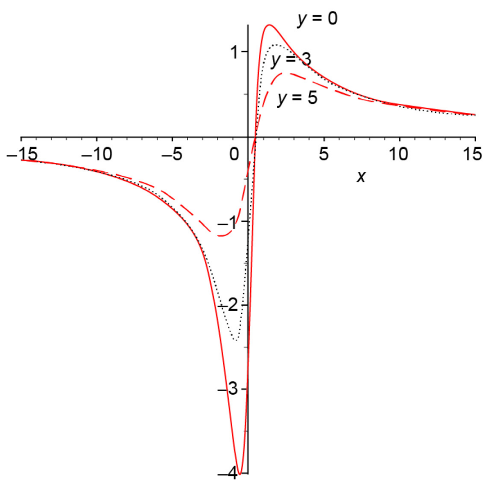

Figure 1

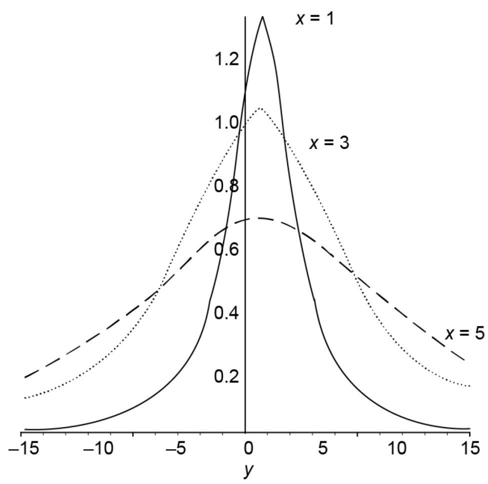

Figure 2

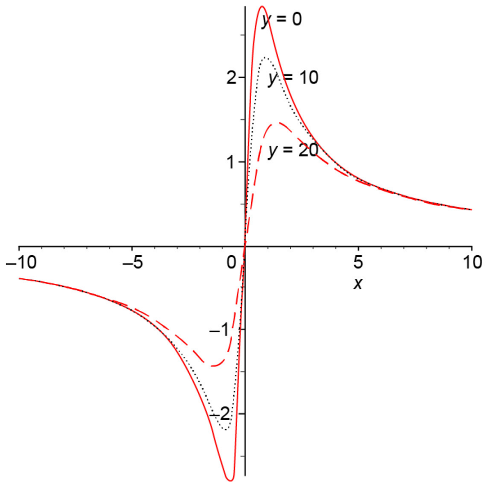

Figure 3

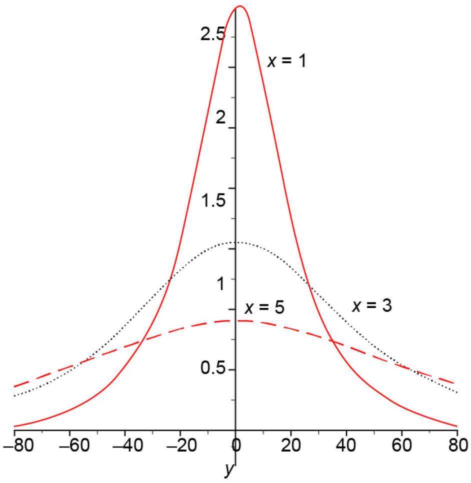

Figure 4

Concluding remarks

We considered a (2+1)-D shallow water wave equation based on its Hirota bilinear forms, and through an intelligent guesswork we put forward the lump solutions of eq. (1). With the help of MAPLE symbolic computation, we presented two special and intuitive pairs of the positive quadratic function solutions.

We remark that it is worth checking if there exists a kind of Wronskian solutions [17, 18] to the (2+1)-D shallow water wave equation, eq. (1). There is also some direct search for rational solutions to non-linear partial differential equations [19, 20], which can be transformed into generalized bilinear equations in terms of generalized bilinear derivatives. Whether those equations have lump solutions would be a very interesting topics for future research.

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