DYNAMICAL ANALYSIS OF LUMP SOLUTION FOR THE (2+1)-DIMENSIONAL ITO EQUATION

by

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Exact kinky breather-wave solution, periodic breather-wave solution, and some lump solutions to the (2+1)-dimensional Ito equation are obtained by using an extended homoclinic test technique and Hirota bi-linear method with a perturbation parameter u_0. Furthermore, a new non-linear phenomenon in the lump solution, is investigated and discussed. These interesting non-linear phenomena might provide us with useful information on the dynamics of higher-dimensional non-linear wave field.

Key words: *(2+1)-dimensional Ito equation, lump solution, dynamics analysis, Hirota bi-linear method*

Introduction

It is well known that non-linear evolution equations appear in many scientific fields: non-linear optics, thermal science, fluid dynamics, solid state physics, *etc.* Studies of these equations have attracted a great deal of attention. Solitary wave solutions of non-linear evolution equations play an important role in non-linear science fields, since they can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications. As a special type of solitary wave, lump solution is a kind of rational function solutions, decayed in all directions in the space. Therefore, the investigation of lump solution for non-linear partial differential equations has become more and more important and attractive. Very recently, lump solutions were presented for many systems [1-6].

In this work, we will try to get lump solutions and analyze its dynamics of the following form:

$$u_{tt} + u_{xxt} + 3(u_x u_t + u u_{xt}) + 3u_{xx} \int_{-\infty}^{x} u_t \mathrm{d}x' + a u_{yt} + b u_{xt} = 0 \tag{1}$$

which is called Ito equation. In 1980, Ito [7] established the well-known (1+1)-dimensional – (1+1)-D, and (2+1)-dimensional – (2+1)-D Ito equation by generalization of the bi-linear KdV equation. Here a, b are arbitrary constants, $u(x, y, t)$ is unknown function of independent variables x, y, t. The (2+1)-D Ito equation is reduced to the (1+1)-D Ito equation when we choose $a = 0$, $b = 0$. In recent years, there are many studies concerning the (2+1)-D Ito equation. Wazwaz obtained its soliton solutions by using the Hirota bi-linear method and tanh-coth method [8].

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Ebadi *et al.* [9] studied its soliton solutions, 1-soliton solution and traveling wave solution using *G/G'* method. Li and Zhao [10] obtained its soliton solutions, periodic solitary wave solutions and doubly-periodic wave solutions using the bi-linear method and the extended homoclinic test technique. Zhao *et al.* [11] used the extend three-wave method to construct its multi-wave solutions, respectively. Tian *et al.* [12] applied the three-soliton limit method with a special parameter to obtain rogue waves and new multi-wave solution of the (2+1)-D Ito equation. However, the lump solution and its dynamics of (2+1)-D Ito equation have not been presented in the previous work [9-12]. In this manuscript, an exact kinky breather-wave solution and periodic breather-wave solution based on bi-linear form are obtained using the extended homoclinic test approach (EHTA) [13], and some lump solutions based on kinky breather-wave and periodic breather-wave are studied by using the homoclinic breather limit method (HBLM) [14]. What is more, we study the dynamic behavior of lump solutions using the extreme value theory of multivariable function, some novel and interesting phenomena are revealed.

Kinky breather-wave and lump solution

In this section, by choosing special test function, an exact kinky breather-wave solution and a lump solution for (2+1)-D Ito equation are obtained using HBLM and Hirota bilinear method. Obviously, an arbitrary constant u_0 is a solution eq. (1). Therefore, by Painleve analysis, we can assume that the solution of eq. (1) takes the form:

$$u(x,y,t) = u_0 + 2(\ln f)_{xx} \tag{2}$$

where $f(x, y, t)$ is an unknown real function which will be determined. Substituting eq. (2) into eq. (1), we can reduce it into the following bi-linear form which is different from the work [12]:

$$D_t[D_t + D_x^3 + aD_y + (b+3u_0)D_x]f\,f = 0 \tag{3}$$

where the *D*- operator is defined by:

$$D_x^m D_t^k a\,b = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right) a(x,t)b(x',t')|_{x'=x,t'=t} \tag{4}$$

In order to get kinky breather-wave solution by using the EHTA, with regard to eq. (3), we choose the test function in the form:

$$f(x,y,t) = \mathrm{e}^{-p\xi} + b_0\cos(p_1\eta) + b_1\mathrm{e}^{p\xi} \tag{5}$$

where $\xi = x + \alpha y + \beta t$, $\eta = x + \alpha_1 y + \beta_1 t$, and α, β, α_1, β_1, p, p_1, b_0, and b_1 are some constants to be determined later. Substituting eq. (5) into eq. (3) and collecting the coefficients of $\mathrm{e}^{-p\xi}$, $\mathrm{e}^{p\xi}$, $\sin(p_1\eta)$, and $\cos(p_1\eta)$, then equating coefficients of these terms to zero, we obtain a set of algebraic equation about α, β, α_1, β_1, p, p_1, b_1, and b_0. From eq. (3), we get:

$$\begin{cases} p^2\beta_1 - p_1^2\beta + 3(p^2\beta - p_1^2\beta_1) + (b+3u_0)(\beta+\beta_1) + a(\beta\alpha_1 + \beta_1\alpha) + 2\beta\beta_1 = 0 \\ p_1^4\beta_1 + p^4\beta + p^2\beta^2 - p_1^2\beta_1^2 - 3p^2p_1^2(\beta+\beta_1) - (b+3u_0)(p_1^2\beta_1 - p^2\beta) + \\ \quad + a(p^2\alpha\beta - p_1^2\alpha_1\beta_1) = 0 \\ b_0^2 p_1^2\beta_1[4p_1^2 - \beta_1 - \alpha\alpha_1 - (b+3u_0)] + 4b_1p^2\beta[\beta + 4p^2 + a\alpha + (b+3u_0)] = 0 \end{cases} \tag{6}$$

Solving the system of eqs. (6) with the aid of MAPLE, we obtain the following sets of solutions:

Case 1

$$\alpha_1 = -\frac{4pb_1\theta + b_0^2 p_1^2 (3p^2 - p_1^2 + 3u_0 + b)}{ab_0^2 p_1^2}, \quad \beta = -\theta, \quad \beta_1 = \frac{4b_1 p^2 \theta}{b_0^2 p_1^2} \tag{7}$$

where $\theta = p^2 + a\alpha_1 + 3u_0 + b - 3p_1^2$ and for some arbitrary real constants α, p, p_1, b_0, and b_1. Substitute eqs. (7) with eq. (5) into eq. (2), we obtain a kinky breather-wave solution, see fig. 1(a):

$$u = u_0 + \frac{2[2\sqrt{b_1}\,p\cosh(\xi + \ln\sqrt{b_1}) - b_0 p^2 \cos\eta]}{2\sqrt{b_1}\cosh(\xi + \ln\sqrt{b_1}) + b_0\cos\eta} - \frac{2[2\sqrt{b_1}\,p\sinh(\xi + \ln\sqrt{b_1}) - b_0 p\sin\eta]^2}{[2\sqrt{b_1}\cosh(\xi + \ln\sqrt{b_1}) + b_0\cos\eta]^2} \tag{8}$$

where $\xi = p(x + \alpha y - \theta t)$ and $\eta = p_1\left[x - \frac{4pb_1\theta + b_0^2 p_1^2(3p^2 - p_1^2 + 3u_0 + b)y}{ab_0^2 p_1^2} + \frac{4b_1 p^2 \theta t}{b_0^2 p_1^2}\right]$

Solution $u(x, y, t)$ represented by eq. (8) is kinky breather-wave solution, the forward-direction (or backward-direction) wave shows breather solitary feature as trajectory along the straight line $x = -(\alpha y + \beta t)$, meanwhile takes on kinky solitary feature as trajectory along the straight line $x = -(\alpha_1 y + \beta_1 t)$ for the (2+1)-D Ito equation. Specially, this wave shows both breather and kinky feature to time, t, and space, x, (or y) [15].

Specially, if we choose $b_0 = -2\cos(p)$, $b_1 = 1$, $p_1 = p$ in the eq. (7) and letting $p \to 0$, we can get the following lump solution, see fig. 1(b):

$$u(x, y, t) = u_0 + \frac{8\left\{1 - 2[x + \alpha y - (a\alpha + b + 3u_0)t]\left[x - \frac{a\alpha + 2b + 3u_0}{a}\,y + (a\alpha + b + u_0)t\right]\right\}}{\left\{1 + [x + \alpha y - (a\alpha + b + 3u_0)t]^2 + \left[x - \frac{a\alpha + 2b + 3u_0}{a}\,y + (a\alpha + b + u_0)t\right]\right\}^2} \tag{9}$$

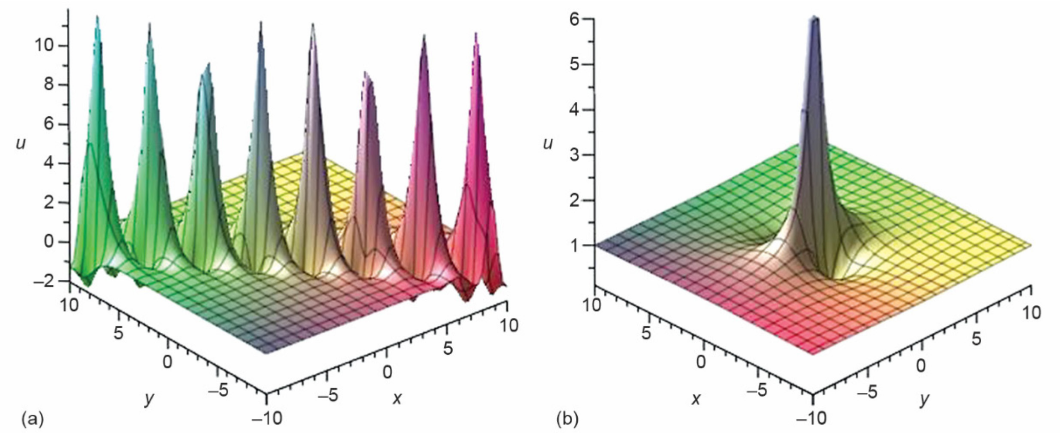

Figure 1. (a) the kinky breather-wave solution as $b_0 = -1$, $p_1 = -2$, $p = \alpha_1 = b_1 = a = b = 1$, $u_0 = 0$, $t = 0$, (b) the lump solution as $\alpha = a = u_0 = 1$, $b = -3$, $t = 0$

Obviously, this solution eq. (9) represents a kind of exact solitary wave solution in the form of the rational solution, this kind of solution is actually called lump solution [3]. This solution contains four free parameters a, b, α, and u_0. The asymptotic behavior of the lump solution eq. (9) can be found $u \to u_0$, either $x \to \pm\infty$, or $y \to \pm\infty$, or $t \to \pm\infty$, it means this solution eq. (9) is also a pulse solution. From fig. 1(b), we can clearly see that the solution eq. (9) has bright lump solution feature. The bright lump solution has one upward peak and two small downward amplitude holes, the main peak forms a much higher hill, the two downward holes are hidden under the plane wave. No matter how these free parameters are changed, there is no dark lump or bright-dark lump structure like the [3].

Now, we focus on the exact lump solution eq. (9) by the perspective of the extreme value theory. Considering the critical point of the function $U(x,y) = u(x, y)$. After calculating, the point $o(x,y) = o(0, 0)$ is a critical point of the function $U(x,y)$. Thus, at the point $o(0, 0)$, the second order derivative can be obtained:

$$\begin{cases} \dfrac{\partial^2}{\partial x^2} U(x,y)\Big|_{o(0,0)} = -96 < 0 \\ H(U) = \begin{vmatrix} \dfrac{\partial^2 U(x,y)}{\partial x^2} & \dfrac{\partial^2 U(x,y)}{\partial x \partial y} \\ \dfrac{\partial^2 U(x,y)}{\partial y \partial x} & \dfrac{\partial^2 U(x,y)}{\partial y^2} \end{vmatrix}_{o(0,0)} = 3072\left(a + \dfrac{b+3u_0}{a}\right)^2 \end{cases} \tag{10}$$

Using the discriminant method of extremum value for the function of two variable, we can the following results: (1) if $a^2 \neq -(b + 3u_0)$, this is, $H(U) > 0$, $o(0, 0)$ is a local maximum point, $U_{\max} = U(0, 0) = 8 + u_0$, $u(x, y, t)$, shows a single bright lump solitary structure, see fig. 1(b) and (2) if $a^2 = -(b + 3u_0)$, $U(x,y) = 0$, the test is inconclusive at $o(0, 0)$ by using the extreme value theory.

Case 2

$$\beta = -(4p^2 + a\alpha + b + 3u_0), \qquad p_1 = ip, \qquad \alpha_1 = -\frac{4p^2 + \beta_1 + b + 3u_0}{a} \tag{11}$$

where α, p, b_0, b_1, and β_1 are some free real constants. Substituting eqs. (11) with eq. (5) into eq. (2), we obtain a two-soliton solution:

$$u = u_0 + \frac{2[2\sqrt{b_1}\,p\cosh(\xi + \ln\sqrt{b_1}) - b_0 p^2 \cosh\eta]}{2\sqrt{b_1}\cosh(\xi + \ln\sqrt{b_1}) + b_0\cosh\eta} - \\ - \frac{2[2\sqrt{b_1}\,p\sinh(\xi + \ln\sqrt{b_1}) - b_0 p\sinh\eta]^2}{[2\sqrt{b_1}\cosh(\xi + \ln\sqrt{b_1}) + b_0\cosh\eta]^2} \tag{12}$$

where $\xi = p[x + \alpha y - (4p^2 + a\alpha + b + 3u_0)t]$ and $\xi = p\{x - [(4p^2 + \beta_1 + b + 3u_0)/a]y + \beta_1 t\}$.

Letting $b_0 = 2\cos(kp)$, $b_1 = 1$, and taking $p \to 0$ in eq. (12), we can get a rational solitary wave solution:

$$u(x,y,t) = u_0 - \frac{8(a\xi - \eta)^2 a^2}{(a^2\xi^2 - \eta^2 + a^2k^2)^2} \tag{13}$$

where $\xi = -x - \alpha y + (a\alpha + b + 3u_0)t$ and $\eta = -ax + (\beta_1 + b + 3u_0)y - a\beta_1 t$.

Periodic breather-wave and new lump solution

In this section, by choosing special test function in application of EHTA to (2+1)-D Ito equation, we obtain a periodic breather-wave solution and a new lump solution by choosing special test function. To obtain periodic breather-wave, we choose the test function in the form:

$$f(x,y,t)=\mathrm{e}^{-p(y+\beta t)}+b_0\cos[p_1(x+\alpha_1 y+\beta_1 t)]+b_1\mathrm{e}^{p(y+\beta t)} \tag{14}$$

where β, α_1, β_1, p, p_1, b_0, and $b_1$are constants to be determined. Substituting eq. (14) into eq. (3), equating all coefficients of different powers of $\mathrm{e}^{j(y+\beta t)}$, ($j$ = –1, 0, 1) to zero, we obtain the following algebraic equations:

$$\begin{cases} a\beta_1-\beta p_1^2+2\beta\beta_1+b\beta+a\beta\alpha_1=0 \\ p_1^4\beta_1-b\beta_1 p_1^2-ap_1^2\alpha_1\beta_1+a\beta p^2-p_1^2\beta_1^2+p^2\beta^2=0 \\ 4ap^2\beta b_1-bb_0^2\beta_1 p_1^2+4b_0^2 p_1^4\beta_1-b_0^2 p_1^2\beta_1^2+4p^2\beta^2 b_1-ab_0^2 p_1^2\alpha_1\beta_1=0 \end{cases} \tag{15}$$

Solving eqs. (15) with MAPLE, we get:

$$\beta_1=0,\qquad \beta=-a,\qquad \alpha_1=a^{-1}[p_1^2-(b+3u_0)] \tag{16}$$

Substituting eq. (16) with eq. (14) into eq. (2), we obtain an exact periodic breather-wave solution of eq. (1), see fig. 2(a):

$$u=u_0-\frac{2b_0p_1^2\left(b_0+\cosh[p(y-at)+\ln\sqrt{b_1}]\cos\left\{p_1\left[x+\dfrac{p_1^2-(b+3u_0)y}{a}\right]\right\}\right)}{\left(2\sqrt{b_1}\cosh[p(y-at)+\ln\sqrt{b_1}]+b_0\cos\left\{p_1\left[x+\dfrac{p_1^2-(b+3u_0)y}{a}\right]\right\}\right)^2} \tag{17}$$

Letting b_0 = –2cos(p), b_1 = 1, p_1 = kp, and taking $p\to 0$ in eq. (17), we can get a new lump solution:

$$u=u_0+\frac{4k^2\left\{1+(y-at)^2-k^2\left[x-\dfrac{(b+3u_0)y}{a}\right]^2\right\}}{\left\{1+(y-at)^2+k^2\left[x-\dfrac{(b+3u_0)y}{a}\right]^2\right\}^2} \tag{18}$$

Similarly, this solution eq. (18) is also a lump solution, which have similar structure and dynamics characteristics with solution eq. (8), see fig. 2(b).

Through the previous theoretical analysis, numerical simulation, and 3-D image simulation, the reason for the dynamic structure of lump solution for the (2+1)-D Ito equation is clearly displayed. The structure of the lump solution is mainly determined the value of the perturbation parameter, u_0, and some free parameters a, b, α, but no matter how these parameters are changed, the lump solution has only one structure: bright lump structure.

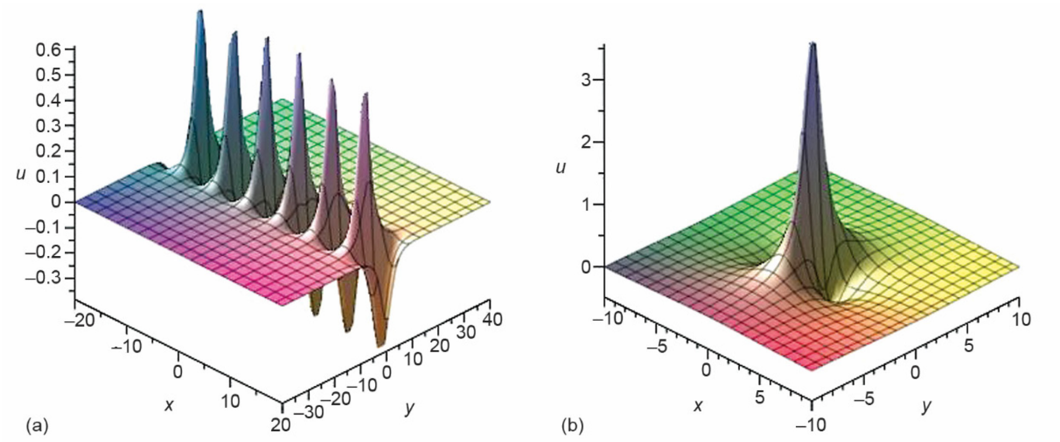

Figure 2. (a) the periodic breather-wave solution as $b_1 = a = b = 1$, $p = b_0 = 0.5$, $p_1 = –1$, $u_0 = t = 0$ and (b) the lump solution as $k = 1$, $u_0 = 1, \cdot a = 0.25$, $b = –3$, $t = 0$

Conclusion

In conclusions, based on the Hirota bi-linear method, extended homoclinic test technique and by using a MAPLE symbolic computation, we obtain a kinky breather-wave solution, see fig. 1(a), a periodic breather-wave solution, fig. 2(a), and some lump solutions to the (2+1)-D Ito equation. From the dynamics analysis of lump solution, this form of the lump solution is only the maximum value – bright lump solution, see figs. 1(b) and 2(b), and has no other forms of solution (dark lump solution and bright-dark lump solution), the maximum of lump solution depends on the perturbation parameters u_0. It is hoped that these results might provide us with useful information on the dynamic fields of the relevant non-linear science. In the future, we will use this method research of other types of non-linear evolution equation.

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