A GENERAL SUB-EQUATION METHOD TO THE BURGERS-LIKE EQUATION

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A Burgers-like equation is studied by a general sub-equation method, and some new exact solutions are obtained, which include the traveling wave solutions, non-traveling wave solutions, multi-soliton solutions, rational solutions, and other types of solutions. The obtained results are important in thermal science, and potential applications can be found.

Key words: *general sub-equation method, Burgers-like equation, exact solution*

Introduction

Many mathematical models of thermal science and fluid mechanics are based on non-linear partial differential equations (PDE). Because of this, an important research area is connected to the obtaining of exact analytical solutions of such equations. In past decades, the most impressive methods for obtaining exact solutions have been presented such as the inverse scattering method [1], Backlund transformation method [2], the homogeneous balance method [3], tanh-function method [4], F-expansion method [5], sub-ordinary differential equation (ODE) method [6], exp-function method [7, 8], (G′/G)-expansion method [9, 10], *etc.*

Recently, we introduced a general sub-equation method to look for exact solution of non-linear evolution equations (NLEE) [11]. In the method, we chose a solution expression in a polynomial form of a solution to a variable coefficient ODE as auxiliary equation. This method can yield a Backlund transformation between the NLEE and a related constraint equation. By dealing with the constraint equation, we can derive infinite number of exact solutions for the NLEE. These solutions include the traveling wave solutions, non-traveling wave solutions, multi-soliton solutions, rational solutions, and other types of solutions.

The exact solutions to the Burgers-like equation

In this article, we give new exact solutions to the Burgers-like equation:

$$u_t + u_x + \beta u u_x - \alpha u_{xx} = 0 \tag{1}$$

where α, and β are constants by applying the general sub-equation method.

The general method

First we show the general procedure of the general sub-equation method. The general sub-equation method is consisted of the following four steps:

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Step 1. By considering the homogeneous balance between the highest order derivatives and non-linear terms in eq. (1), we suppose that the solutions to eq. (1) can be expressed in the following general form:

$$u(t,x) = a_0(t,x) + a_1(t,x)\phi(\xi) \tag{2}$$

where functions $a_0(t, x)$, $a_1(t, x)$, and $\xi = \xi(t, x)$ are determined later, and $\phi = \phi(\xi)$ satisfies a first order ODE:

$$\xi^2(\phi' + \phi^2) + \delta\xi\phi + \mu = 0 \tag{3}$$

It has three types of general solution:

$$\phi_1(\xi) = \varepsilon \times \frac{[(1-\delta)c_1 - 2\eta c_2]\sin(\eta \ln|\xi|) + [2\eta c_1 + (1-\delta)c_2]\cos(\eta \ln|\xi|)}{2\xi[c_1 \sin(\eta \ln|\xi|) + c_2 \cos(\eta \ln|\xi|)]}, \quad \text{if } s > 1 \tag{4}$$

$$\phi_2(\xi) = \varepsilon \times \frac{c_1(2\eta - \delta + 1)|\xi|^{2\eta} - c_2(2\eta + \delta - 1)}{2|\xi|(c_1|\xi|^{2\eta} + c_2)}, \quad \text{if } s < 1 \tag{5}$$

$$\phi_3(\xi) = \varepsilon \times \frac{(1-\delta)(c_1 + c_2 \ln|\xi|) + 2c_2}{2\xi(c_1 + c_2 \ln|\xi|)}, \quad \text{if } s = 1 \tag{6}$$

where c_1, c_2, δ, and μ are constants,

$$\varepsilon = \begin{cases} 1, & \text{if } \xi > 0 \\ -1, & \text{if } \xi < 0 \end{cases}$$

and

$$s = 2\delta - \delta^2 + 4\mu, \quad \eta = \frac{1}{2}|1-s|^{\frac{1}{2}} \tag{7}$$

Step 2. We substitute eq. (2) along with the auxiliary eq. (3) into eq. (1), and collect all terms with the same order of ϕ. As a result, the left-hand sides of eq. (1) are converted into a polynomials in ϕ. Equating each coefficient of powers of ϕ to zero, we obtain a set of over-determined PDE for $a_0(t, x)$, $a_1(t, x)$, and $\xi(t, x)$.

Step 3. Solving the differential system obtained in *Step 2* by MATHEMATICA, we obtain the expressions of $a_0(t, x)$, $a_1(t, x)$ represented by $\xi(t, x)$:

$$a_0(t,x) = (\alpha\xi\xi_{xx} - \xi\xi_t - \xi\xi_x - \alpha\delta\xi_x^2)(\beta\xi\xi_{xx})^{-1}, \quad a_1(t,x) = -2\alpha\beta^{-1}\xi_x \tag{8}$$

where $\xi = \xi(t, x)$ satisfies the equation:

$$\alpha^2 s(\xi_x^2 - \xi\xi_{xx})\xi_x^4 + \xi^3[(\alpha\xi_{xxxx} - 2\xi_{txx} + \xi_{tt})\xi_x^2 + (4\xi_{xx} - 2\xi_t)(\xi_{tx} - \alpha\xi_{xxx})\xi_x +$$
$$+ (\xi_t^2 + 3\alpha^2\xi_{xx}^2 - 4\alpha\xi_t\xi_{xx})\xi_{xx}] = 0 \tag{9}$$

Equation (9) is called *the constraint equation* of ξ.

For obtaining non-trivial solutions, we consider the case of $\xi_x \neq 0$. Any solution of eq. (9) leads to a group of coefficients in eq. (8). Consequently, we obtain three classes of exact solutions (2) of eq. (1):

$$u_i(t,x) = a_0(t,x) + a_1(t,x)\phi_i(\xi) \tag{10}$$

where $i = 1$ for $s > 1$, $i = 2$ for $s < 1$, and $i = 3$, for $s = 1$. The $\phi_i(\xi)$ is the solutions of eq. (3).

Therefore, the solution expressions (10) have established a Backlund transformation between eq. (1) and eq. (9). By using the Backlund transformation, one can obtain infinite number of exact solutions to eq. (1). When $\xi = \xi(x - Vt)$ is a solution of eq. (9), we can obtain the exact traveling wave solutions of eq. (1). If $\xi \neq \xi(x - Vt)$, we can obtain the exact non-traveling wave solutions of eq. (1). So the solution of expression (10) provide us with abundance of general form non-trivial exact solutions to eq. (1) when the solutions of eq. (9) are given.

Step 4. In the following, we determine the exact traveling wave solutions and non--traveling wave solutions of eq. (1) by solving eq. (9).

The exact traveling wave solutions

If $\xi(t, x) = \xi(x - Vt)$ then eq. (9) can become the following form:

$$s[\xi'(z)^2 - \xi(z)\xi''(z)]\xi'(z)^4 + [4\xi'(z)\xi''(z)\xi^{(3)}(z) - \xi'(z)^2\xi^{(4)}(z) - 3\xi''(z)^3]\xi(z)^3 = 0 \tag{11}$$

where $z = x - Vt$. With the transformation $\xi(t, x) = \xi(z)$, $\xi'(z) = Y(\xi)$, eq. (11) becomes:

$$\xi^3 Y^{(3)}(\xi) + s[\xi Y'(\xi) - Y(\xi)] = 0$$

which has general solutions:

$$Y(\xi) = \begin{cases} |\xi|[d_1 + d_2\cos(\delta_0 \ln|\xi|) + d_3\sin(\delta_0 \ln|\xi|)], & \text{if } \delta_0^2 = s - 1 > 0 \\ |\xi|(d_1|\xi|^{-\delta_0} + d_2|\xi|^{\delta_0} + d_3), & \text{if } \delta_0^2 = 1 - s > 0 \\ |\xi|(d_1 + d_2\ln|\xi| + d_3\ln^2|\xi|), & \text{if } \delta_0^2 = 1 - s = 0 \end{cases} \tag{12}$$

where d_1, d_2, and d_3 are arbitrary constants. For a non-trivial solution, these constants should not be equal to zero simultaneously. Correspondingly, from $\xi'(z) = Y(\xi)$ and eq. (12), we have obtained twelve kinds of exact solutions to eq. (9) [11]. Substituting these solutions into eqs. (4)-(6) and (8), and assembling them in eq. (10), we can obtain twelve kinds of exact traveling wave solutions of eq. (1). The multi-composite solutions which are constructed by compounding several elementary functions is a feature of these solutions. For example, in the case $s > 1$, the solutions are the compounding of five elementary functions exp, arctan, tanh, tan, arcos, *etc.* These solutions can not be obtained by using other methods [4-10]. When $s < 1$ or $s = 1$, these solutions have included soliton solutions and rational or irrational solutions if we properly take the parameters. These solutions already significantly expand the set of exact solutions to eq. (1). Here we omit showing of the expressions of these solutions because of space.

New exact solutions of the Burgers-like equation

In fact, eq. (9) can be written:

$$s(\xi\xi_{xx} - \xi_x^2)\xi_x^4 + \xi^3[P(P - 2\alpha\xi_{xx})\xi_{xx} + 2P_x(\alpha\xi_{xx} - P)\xi_x + (P_t - \gamma P_x - \alpha P_{xx})\xi_x^2] = 0 \tag{13}$$

where $P = \xi_t - \alpha\xi_{xx} - \gamma\xi_x$ for arbitrary constant γ.

We consider the following two cases for getting more general form exact solutions to eq. (9).

Case I. For $\xi\xi_{xx} - \xi_x^2 = 0$ and $s = 2\delta - \delta^2 + 4\mu$.

In this case, we obtain $\xi(t, x) = d_4 \exp[(d_2 x + d_3)(t + d_1)^{-1}]$, which results in exact solutions of eq. (1) given by (10) with (8):

$$u_i(t,x) = \frac{d_3 + d_2[x - t - d_1 + d_2\alpha(1-\delta)]}{d_2(d_1+t)\beta} - \frac{2\alpha d_2 d_4}{(d_1+t)\beta}\exp\left(\frac{d_3 + d_2 x}{d_1 + t}\right)\phi_i(\xi) \tag{14}$$

in which $i = 1$ for $s > 1$, $i = 2$ for $s < 1$, and $i = 3$ for $s = 1$. The $\phi_i(\xi)$ is the solutions of eq. (3), and d_i are arbitrary constants.

Case II. For $s = 2\delta - \delta^2 + 4\mu = 0$.

In this case, eqs. (9) or (13) admits the solutions $\xi(t, x)$ of the linear heat equation:

$$\xi_t = \alpha\xi_{xx} + \gamma\xi_x \tag{15}$$

with a conductive term $\gamma\xi_x$ for arbitrary constant γ.

We have found a transformation in the form of eq. (10) between eqs. (1) and (15). Using the transformation, we will obtain infinite number of exact solutions of eq. (1). That means each solution of eq. (15) yields a set of exact solutions of eq. (1). Moreover, the connection provides us with an insight into integrability of eq. (1). For example, we can study the multi-soliton solutions, rational solutions and other types of solutions to the Burgers-likes equation by the transformation.

As we know, eq. (15) has infinite number of exact solutions. Some representatives of them are listed:

$$\xi(t,x) = \frac{A}{\sqrt{t}}\exp\left[-\frac{(x+\gamma t)^2}{4\alpha t}\right] + B, \quad (t,x) \in R^2$$

$$\xi(t,x) = A\exp(-\alpha k^2 t)\cos[k(x+\gamma t) + B] + C, \quad (t,x) \in R^2$$

$$\xi(t,x) = A\exp[-k(x+\gamma t)]\cos[kx + k(\gamma - 2\alpha k)t + B] + C, \quad (t,x) \in R^2$$

$$\xi(t,x) = c_0 \,\mathrm{erfc}\left(\frac{x+\gamma t}{2\sqrt{\alpha t}}\right), \quad t > 0, \quad x + \gamma t > 0, \quad \alpha > 0 \tag{16}$$

$$\xi(t,x) = \sum_{n=1}^{+\infty} c_n \sin\left[\frac{n\pi(x+\gamma t)}{L}\right]\exp\left[-\tau\left(\frac{n\pi}{L}\right)^2 t\right], \quad t > 0, \quad 0 \le x + \gamma t \le L, \quad L > 0$$

$$\xi(t,x) = \frac{1}{2\sqrt{\pi\alpha t}}\int_{-\infty}^{+\infty}\omega(\theta)\exp\left[-\frac{(x+\gamma t-\theta)^2}{4\tau t}\right]\mathrm{d}\theta, \quad t > 0, \quad x \in R, \quad \alpha > 0$$

where A, B, C, c_0, and k are the arbitrary constants, $\omega(\theta)$ – the any integrable function, erfc(y) – the error function, and c_n, $n = 1, 2,\ldots$ is constant given by the initial or boundary conditions of eq. (15). It is noticed that the last solution in eq. (16) yields an exact solution to the Burgers-like equation containing an arbitrary function, which may give more freedom to solve related problems of the equations.

General form exact solutions

In *Case II*, substituting any solution of eq. (15) into eq. (8) and assembling them in eq. (10) with $i = 2$, we can obtain exact solutions to eq. (1). We give four exact solutions u_j ($j = 1, 2, 3, 4$) of eq. (1):

$$\xi_1(t,x) = At^{-1/2}\exp[-(x+\gamma t)^2(4\alpha t)^{-1}] + B, \quad (t,x)\in R^2$$

$$u_1(t,x) = \frac{-2Bt^{3/2}(1+\gamma) + Ah_1(t,x)[\gamma t(\delta-2) - 2t + \delta x]}{2\beta[Ah_1(t,x)+Bt^{1/2}]t} + A\beta^{-1}t^{-3/2}h_1(t,x)(x+\gamma t)\phi_2(\xi_1) \quad (17)$$

$$\xi_2(t,x) = A\exp(-\alpha k^2 t)\cos[k(x+\gamma t)+B] + C, \quad (t,x)\in R^2$$

$$u_2(t,x) = \frac{-C(1+\gamma)h_2(t,x) - A(1+\gamma)\cos\eta_0 + Ak\alpha\delta\sin\eta_0}{\beta[Ch_2(t,x) + A\cos\eta_0]} + \frac{2Ak\alpha\sin\eta_0}{\beta h_2(t,x)}\phi_2(\xi_2) \quad (18)$$

$$\xi_3(t,x) = A\exp[-k(x+\gamma t)]\cos(kx + \gamma kt - 2\alpha k^2 t + B) + C, \quad (t,x)\in R^2$$

$$u_3(t,x) = \frac{Ak\alpha\delta\sin\eta_1 - C(1+\gamma)h_3(t,x) - A(1+\gamma-k\alpha\delta)\cos\eta_1}{\beta[Ch_3(t,x) + A\cos\eta_1]} + \frac{2Ak\alpha h_4(t,x)}{\beta h_3(t,x)}\phi_2(\xi_3) \quad (19)$$

$$\xi_4(t,x) = c_0\,\mathrm{erfc}[2^{-1}(x+\gamma t)(\alpha t)^{-1/2}], \quad t>0, \quad x+\gamma t>0, \quad \alpha>0$$

$$u_4(t,x) = -\frac{1+\gamma}{\beta} + \frac{\alpha\delta h_1(t,x)}{\beta\sqrt{\pi\alpha t}\,\mathrm{erfc}(\eta_2)} + \frac{2c_0\alpha h_1(t,x)}{\beta\sqrt{\pi\alpha t}}\phi_2(\xi_4) \quad (20)$$

In all of these cases, $\phi_2(\xi)$ is given in eq. (5), and A, B, C, and c_0 are arbitrary constants,

$$\eta_0 = B + k(x+\gamma t), \quad \eta_1 = B + kx + \alpha kt(1-2k), \quad \eta_2 = 2^{-1}(x+\gamma t)(\alpha t)^{-1/2},$$

$$h_1(t,x) = \exp\left[-\frac{(x+\gamma t)^2}{4\alpha t}\right], \quad h_2(t,x) = \exp(\alpha k^2 t), \quad h_3(t,x) = \exp[k(x+\gamma t)],$$

$$h_4(t,x) = \cos\eta_1 + \sin\eta_1$$

By the same manner, from the rest two solutions in eq. (16), we should obtain additional exact solutions in different styles to the Burgers-like equation. Due to the lack of space, we omit the reasoning and the solution expressions.

Remark. In author's knowledge, this is the first time showing these kinds of exact solutions to eq. (1).

The multi-soliton solutions

Using the transformation between solutions of eqs. (1) and (15), we can get the multi-soliton solutions of eq. (1). We can check that eq. (15) admits solutions:

$$\xi(t,x) = A_0 + \sum_{i=1}^{n} A_i\exp[k_i x + (\alpha k_i^2 + \gamma k_i)t + B_i]$$

for arbitrary constants k_i, A_i, B_i (i = 0, 1,…,n) and positive integer n, which yield multi-soliton solutions of eq. (1) given by eqs. (8) and (10) with i = 2.

The rational solutions

Suppose that eq. (15) has a solution in the form:

$$\xi(t,x)=\sum_{i=0}^{n}k_i(X)t^i,\quad X=x+\gamma t \tag{21}$$

for arbitrary positive integer n. Substituting eq. (21) into eq. (15) and setting the coefficients of $t^i(i$ = 0, 1,… n) to be zero, we obtain ODE for $k_i(X)$ as $k_n''=0$ and $\alpha k_{j-1}''=k_j, j$ = n, …, 1 which yield the solutions:

$$k_i(X)=\sum_{j=1}^{2(n+1-i)}\alpha_j\frac{X^{2(n+1-i)-j}}{[2(n+1-i)-j]!},\quad i=0,1,2,\cdots,n$$

for arbitrary constants α_j. Substituting the solutions into eq. (21), we obtain the polynomial solutions of eq. (15), which yield rational solutions of eq. (1) given by eqs. (8) and (10) with i = 2.

Conclusions

In this paper, a great number of new exact solutions of the Burgers-like equation are obtained by using the general sub-equation method. Essentially, the constraint equation is related to linear heat equation which results in new exact solutions of the Burgers-like equation. These solutions include the traveling wave solutions, non-traveling wave solutions, multi-soliton solutions, rational solutions, and other types of solutions. These solutions are different from those given in [12]. It shows that our method is more flexible in finding more general form exact solutions and the method can be used for many other NLEE in mathematical physics. As a result, our method effectively enhances the existing auxiliary equation methods, such as those given in [4, 9], which demonstrates that the proposed method is effective and prospective. The obtained results are important in thermal science, and can be found potential applications.

We have also tested other several commonly used auxiliary equations, such as Riccati equation [4], the auxiliary equation of F-expansion method [5], to get exact solutions of the Burgers-like equation considered in the article. But, it can not produce the results presented in the paper. So eq. (3) has its own advantages which are not admitted by other auxiliary equations.

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