ANALYTICAL SOLUTIONS OF DIFFERENTIAL-DIFFERENCE SINE-GORDON EQUATION

by

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In modern textile engineering, non-linear differential-difference equations are often used to describe some phenomena arising in heat/electron conduction and flow in carbon nanotubes. In this paper, we extend the variable coefficient Jacobian elliptic function method to solve non-linear differential-difference sine-Gordon equation by introducing a negative power and some variable coefficients in the ansatz, and derive two series of Jacobian elliptic function solutions. When the modulus of Jacobian elliptic function approaches to 1, some solutions can degenerate into some known solutions in the literature.

Key words: *differential-difference sine-Gordon equation, analytical solutions,*
Jacobian elliptic function method

Introduction

In modern textile engineering, non-linear differential-difference equations (DDE) are often used to describe some phenomena arising in heat/electron conduction and flow in carbon nanotubes. In light of E-infinity theory [1], space at the quantum scale is not a continuum, and it is clear that nanotechnology possesses a considerable abundance which bridges the gap between the discrete and the continuum [2].

On nanoscales, He *et al.* [3] found experimentally an uncertainty phenomenon similar to Heisenberg's uncertainty principle in quantum mechanics. Thus, continuum hypothesis on the nanoscales becomes invalid. Just as the viewpoint of Wu *et al.* [4], it is only approximately true that permeability and capillary pressure in-plane infiltration of assemblies of woven fabric are generally derived using the continuum model. The governing equations for non-continuum media are described by DDE. Moreover, He and Zhu [5] also earlier suggested some DDE describing fascinating phenomena arising in heat/electron conduction and flow in carbon nanotubes. Therefore, the investigation of exact solutions of DDE plays a crucial role in the modelling of modern textile engineering. Unlike difference equations which are fully discretized, DDE are semi-discretized with some (or all) of their spatial variables discretized while time is usually kept continuous.

To our knowledge, exact solutions of continuous non-linear partial differential equations have been extensively studied [6-12]. However, less work has been done to study exact solutions of DDE because it is difficult to find the iterative relations between lattice indices, that is, from indices n to $n \pm 1$. Recently, one generalizes many methods for continuous PDE to solve DDE [13-16]. However, the variable coefficient methods are less extended to solve DDE.

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In this paper, we extend the Jacobian elliptic function method [15] to a variable co-efficient method, and use this method to solve the discrete sine-Gordon equation [14, 16]:

$$\frac{du_{n+1}}{dt} - \frac{du_n}{dt} = \sin(u_{n+1} + u_n) \tag{1}$$

whose space is discrete, with lattice label n, and time is continuous. The sine-Gordon equation has served repeatedly as a prototype for a 2-D non-linear field theory. This equation firstly appeared in the propagation of fluxons in Josephson junctions between two superconductors, then in many scientific fields such as differential geometry, solid-state physics and disloca-tions in metals. Pilloni and Levi [17] developed, in a concise way, the generalization of the discrete Zakharov and Shabat spectral transform necessary to solve the discrete sine-Gordon equation

Mathematical approach

In many cases [13-16], it is difficult to directly solve DDE and some transformations should be introduced, thus we firstly introduce the following two different transformations.

At first, we introduce the transformation:

$$v_n = e^{iu_n}, \qquad v_{n+1} = e^{iu_{n+1}} \tag{2}$$

thus eq. (1) changes into a polynomial-type equation:

$$2v_n \frac{dv_{n+1}}{dt} - 2v_{n+1} \frac{dv_n}{dt} - v_n^2 v_{n+1}^2 + 1 = 0 \tag{3}$$

Now we assume that eq. (3) has the following ansatz:

$$v_n = \sum_{i=-l}^{l} a_i(t) sn^i(\xi_n) \tag{4}$$

Via the analysis of the leading term, we have $l = 1$, and thus:

$$v_{n\pm1} = a_0(t) + a_1(t)\frac{sn(\xi_n)cn(k)dn(k) \pm sn(k)cn(\xi_n)dn(\xi_n)}{1 - m^2 sn^2(\xi_n)sn^2(k)} +$$

$$+a_{-1}(t)\frac{1 - m^2 sn^2(\xi_n)sn^2(k)}{sn(\xi_n)cn(k)dn(k) \pm sn(k)cn(\xi_n)dn(\xi_n)} \tag{5}$$

where $\xi_n = k(t)n + c(t)$ and the Jacobian elliptic functions $sn(\cdot) \equiv sn(\cdot, m)$, $cn(\cdot) \equiv cn(\cdot, m)$, $dn(\cdot) \equiv dn(\cdot, m)$ with the modulus $m(0 < m < 1)$ [15]. Note that comparing with the Jacobian elliptic function method in [15], here we introduce the negative power, *i. e.* $sn^{-1}(\xi_n)$, and vari-able parameters, *i. e.* $a_i(t)$, in ansatz (5).

Inserting eqs. (4) and (5) into eq. (3), clearing the denominator and eliminating the coefficients of all powers like $sn^i(\xi_n)$ and $cn(\xi_n)dn(\xi_n)sn^j(\xi_n)$, one obtains:

$$a_0 = b_1 = 0, \quad k = d, \quad c(t) = \frac{\pm t}{2msn(d)} + c_0, \quad a_1 = \pm\frac{1}{\sqrt{2sn(d)\frac{dc}{dt}}} \tag{6}$$

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or

$$a_0 = a_1 = 0, \quad k = d, \quad c(t) = \frac{\pm t}{2msn(d)} + c_0, \quad b_1 = \pm \frac{1}{\sqrt{2sn(d)\dfrac{dc}{dt}}} \tag{7}$$

or

$$a_0 = 0, \quad k = d, \quad c(t) = \pm\sqrt{\frac{1}{m}}\frac{[1 + msn^2(d)]t}{4(m+1)sn(d)} + c_0, \quad a_1 = \frac{b_1}{m} = \pm \frac{m\sqrt{sn(d)\dfrac{dc}{dt}}}{\sqrt{2msn^2(d) + 2}} \tag{8}$$

where d is a real constant. Therefore, the exact travelling wave solutions of the discrete sine--Gordon eq. (1) has the form:

$$u_n = \arccos \frac{\pm\sqrt{m}[sn(\xi_n) + ns(\xi_n)]}{2} \tag{9}$$

with $\xi_n = dn \pm t/[2msn(d)] + c_0$ and a real constant d, and:

$$u_n = \arccos\left\{ \frac{\pm[m(m+1)^2]^{-1/4}[msn(\xi_n)]}{2} \pm \frac{[m(m+1)^2]^{1/4}[msn(\xi_n) + ns(\xi_n)^{-1}]}{2} \right\} \tag{10}$$

with $\xi_n = dn \pm (1/m)^{1/2}\{[1 + msn^2(d)]\,t\}/[4(m+1)sn(d)] + c_0$. When the modulus $m \to 1$, solutions (9) and (10) are:

$$u_n = \arccos \frac{\pm[\tanh(\xi_n) + \coth(\xi_n)]}{2} \tag{11}$$

with $\xi_n = dn \pm t/[2\tanh(d)] + c_0$, and:

$$u_n = \arccos\left\{ \frac{\pm[\tanh(\xi_n) + \coth(\xi_n)]}{2\sqrt{2}} \pm \frac{[\tanh(\xi_n) + \coth(\xi_n)]^{-1}}{\sqrt{2}} \right\} \tag{12}$$

with $\xi_n = dn \pm [1 + \tanh^2(d)]/[8\tanh(d)]t + c_0$. Here, solution (12) is the corresponding solution in ref. [15].

Next we introduce the transformation:

$$u_n = 2\arctan(v_n), \qquad u_{n+1} = 2\arctan(v_{n+1}) \tag{13}$$

and hence eq. (1) is reduced to a polynomial-type equation:

$$(1 + v_n^2)\frac{dv_{n+1}}{dt} - (1 + v_{n+1}^2)\frac{dv_n}{dt} - v_n(1 - v_{n+1}^2) - v_{n+1}(1 - v_n^2) = 0 \tag{14}$$

The balance procedure admits us to assume that eq. (14) has the ansatz (4) with $i = 1$. Along the similar procedure, we obtain exact travelling wave solution of eq. (1):

$$u_n = 2\arctan\left\{ \pm sn\left[dn - \frac{sn(d)}{1 - cn(d)dn(d) + msn^2(d)}t + c_0 \right] \right\} \tag{15}$$

If the modulus $m \to 1$, solution (15) degenerates into:

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$$u_n = 2\arctan\left\{\pm\tanh\left[dn - \frac{1}{2}\cot(d)t + c_0\right]\right\} \tag{16}$$

which is the corresponding solution in [15].

Results

It is well known that periodic solutions and soliton solutions are interesting and physical relevance. In the following, we take solutions (15) and (16) as examples to further analyze their dynamical properties by some figures.

Figure 1(a) displays the evolutional behavior of solution (15) with $m = 0.99$ in the lattice *vs.* the time co-ordinate. For the fixed time, t, the travelling wave in different lattice n has different amplitude with a periodic variation. For the fixed lattice n, the travelling wave oscillates periodically with the increasing time, t.

Figure 1(b) shows a sectional view of dynamical behavior of solution (15) at time $t = 10$ with different modulus numbers m. Red diamond and black circle line denote $m = 0.95$ and $m = 0.99$, respectively. From it, one can see that the period of travelling wave adds for +-branch and – -branch of solutions (15) and (16), respectively.

In fig. 1(c), black circle line and blue cross line denote +-branch and – -branch of solution (15), respectively. In fig. 1(d), black circle and red diamond line denote +-branch and – -branch of solution (16), respectively. From them, we find that the plus and minus signs have different influence on the periodic solution (15) and soliton solution (16). The periodic solution (15) with plus sign (+-branch) or minus sign (– -branch) has the same wave shape, but exists a phase shift. The soliton solution (16) with plus sign (+-branch) or minus sign (– -branch) has different wave shape, whose difference lies in kink and anti-kink wave shape.
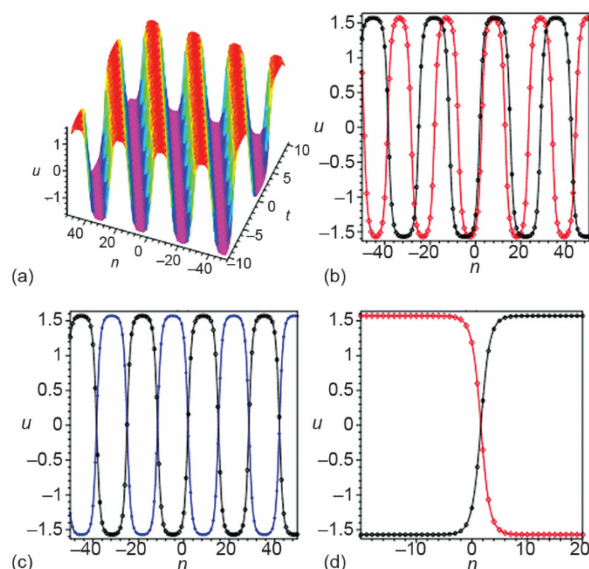

Figure 1. (a) Dynamical behavior of solution (15) with *m* = 0.99 in the lattice *vs.* the time co-ordinate, (b) sectional view corresponding to (a) at time *t* = 10 with *m* = 0.95 (red diamond) and *m* = 0.99 (black circle), (c) sectional view of dynamical behavior for solution (15) at time *t* = 10 for *m* = 0.99 with +-branch (black circle) and – -branch (blue cross), and (d) sectional view of dynamical behavior for solution (16) at time *t* = 10 with +-branch (black circle) and – -branch (red diamond). The parameters are chosen as *d* = 0.5, *c₀* = 10
(*for color image see journal web site*)

Conclusion

In conclusion, we have utilized the variable coefficient Jacobian elliptic function method to construct two series of exact travelling solutions for the discrete sine-Gordon equa-

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tion. When the modulus $m \rightarrow 1$, some solutions can degenerate into some known solutions [15]. Of course, this method presented in this paper is only an initial work, more work about the method should be concerned. For example, extend this method to higher dimensional systems and a set of coupled equations. Therefore, more applications in the modern textile engineering deserve further investigation.

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