THE ADOMIAN DECOMPOSITION METHOD AND THE FRACTIONAL COMPLEX TRANSFORM FOR FRACTIONAL BRATU-TYPE EQUATION

by

Huan-Huan WANG [a]***, Yue HU*** [a*]***, and Kang-Le WANG*** [b]

[a] School of Mathematics and information Science, Henan Polytechnic University, Jiaozuo, China
[b] School of Mathematics and Statistics, Xidian University, Xi'an, China

Original scientific paper
https://doi.org/10.2298/TSCI160805058W

In this paper, the Adomian decomposition method and the fractional complex transform are adopted to solve a fractional Bratu-type equations based on He's fractional derivative. The solution process is elucidated and analytical results can be directly used in practical applications.

Key words: *He's fractional derivative, fractional complex transform, Adomian decomposition method, fractional Bratu-type equation*

Introduction

Bratu-like equations arising in electrospinning and bubble electrospinning processes have caught much attention [1-3]. The model was first proposed by Wan *et al.* [3] for classic electrospinning, and the model is called Wan model [4], which can also be applied to the bubble electrospinning. The model can describe the morphology of nanofibers well [5]. However, a fractional modification of Wan model will be much more suitable for description of the spinning process.

The fractional differential equations have gained considerable attention of physicists, mathematicians, and engineers in the past two decades [6-8]. With the help of fractional derivatives, all kinds of interdisciplinary problems can be modeled such as signal processing, fluid mechanics, dynamic of viscoelastic material, and statistical mechanics, *etc.* [9, 10]. In general, most fractional differential equations are very difficult to find their exact solutions, so numerical and approximation techniques have to be used. These methods include, the homotopy perturbation method [11-14], variational iteration method [15, 16], exp-function method [17], sub-equation method [18], and others. In this paper, Adomian decomposition method (ADM) [19, 20] and He's fractional complex transform [21, 22] are used to solve fractional Bratu's initial value problem:

$$\begin{cases} \mathrm{D}_x^{2\alpha} u(x) + \lambda \mathrm{e}^{u(x)} = 0 \\ 0 < \alpha \le 1, \quad 0 < x < 1 \\ u(0) = u^{(\alpha)}(0) = 0 \end{cases} \tag{1}$$

* Corresponding author, e-mail: huu3y9@163.com

where λ is a constant, its definition was given in [23], and α is an order of He's fractional derivative defined [24, 25]:

$$\mathrm{D}_t^{\alpha} u = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1} [u_0(s) - u(s)] \mathrm{d}s \tag{2}$$

where $u_0(x, t)$ is the solution of its continuous partner of the problem with the same initial condition of the fractal partner.

Adomian decomposition method

To illustrate ADM, we define the unknown function $u(x)$:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{3}$$

where the components $u_n(x)$ are usually determined recurrently. The non-linear operator $F(u)$ can be decomposed into the following result:

$$F(u) = \sum_{n=0}^{\infty} A_n \tag{4}$$

where A_n are called Adomian polynomials of u_0, u_1, $u_2 \ldots u_n$ given by:

$$A_n = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \left[F\left(\sum_{i=0}^{n} \lambda^i u_i \right) \right]_{\lambda=0}, \qquad n = 0,1,2,3... \tag{5}$$

or equivalently:

$$A_0 = F(u_0), \qquad A_1 = u_1 F'(u_0), \qquad A_2 = u_2 F'(u_0) + \frac{1}{2} u_1^2 F''(u_0),$$
$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3} u_1^3 F'''(u_0)$$

These polynomials can be generated for all classes of non-linearity by using eq. (5). Recently, an alternative algorithm for constructing Adomian polynomials has been proposed by Wazwaz [19].

Application

In this section, one example on fractional Bratu-type equation is solved to demonstrate the performance and efficiency of our proposed method with He's fractional derivative.

Consider fractional Bratu-type equation with initial condition:

$$\mathrm{D}_x^{2\alpha} u(x) - \pi^2 \mathrm{e}^{u(x)} = 0, \qquad 0 < \alpha \le 1, \qquad 0 < x < 1, \qquad u(0) = 0, \qquad u^{(\alpha)}(0) = \pi \tag{6}$$

According to the fractional complex transform:

$$X = \frac{x^{\alpha}}{\Gamma(1+\alpha)} \tag{7}$$

Equation (6) is converted to a partial differential equation, which reads:

$$\mathrm{D}_X^2 u(X) - \pi^2 \mathrm{e}^{u(X)} = 0, \qquad u(0) = 0, \qquad u'(0) = \pi \tag{8}$$

Equation (8) can be written in an operator form:

$$Lu = \pi^2 e^u, \qquad u(0) = 0 \tag{9}$$

where the differential operator L is defined:

$$L = \frac{\partial^2}{\partial X^2} \tag{10}$$

The inverse operator of L is given by:

$$L^{-1}(\cdot) = \int_0^X \int_0^X (\cdot)\mathrm{d}X\mathrm{d}X \tag{11}$$

Using the inverse operator L^{-1} on both sides of eq. (8) and applying the $u(0) = 0$, $u'(0) = \pi$, we have:

$$u(X) = \pi X + L^{-1}(\pi^2 e^u) \tag{12}$$

Substituting eqs. (3) and (4) into the eq. (9), we obtain:

$$\sum_{n=0}^{\infty} u_n(X) = \pi X + L^{-1}(\pi^2 \sum_{n=0}^{\infty} A_n) \tag{13}$$

where A_n are the Adomian polynomials. Identifying the zeroth component $u_0(X) = \pi X$, the remaining components can be obtain by applying the iteration relation:

$$u_{k+1}(X) = \pi^2 L^{-1}(A_k), \qquad k \ge 0 \tag{14}$$

where A_k are Adomian polynomials :

$$A_0 = e^{u_0}, \qquad A_1 = u_1 e^{u_0}, \qquad A_2 = \left(u_2 + \frac{1}{2}u_1^2\right)e^{u_0}, \qquad A_3 = \left(u_3 + u_1 u_2 + \frac{1}{6}u_1^3\right)e^{u_0},$$

$$A_4 = \left(u_4 + u_1 u_3 + \frac{1}{2}u_2^2 + \frac{1}{2}u_1^2 u_2 + \frac{1}{24}u_1^4\right)e^{u_0}$$

By iteration, we have the following results:

$$u_0(X) = \pi X, \qquad u_1(X) = e^{\pi X} - \pi X - 1, \qquad u_2(X) = -\frac{1}{4}(-e^{2\pi X} + 4\pi X e^{\pi X} - 4e^{\pi X} + 2\pi X + 5),$$

$$u_3(X) = \frac{1}{12}(e^{3\pi X} + 6e^{2\pi X}(1 - \pi X) + 3e^{\pi X}(2\pi^2 X^2 - 6\pi X + 5) - 6\pi X - 22)$$

The approximate solution of eq. (8) is given:

$$u(X) = \pi X + (e^{\pi X} - \pi X - 1) - \frac{1}{4}(-e^{2\pi X} + 4\pi X e^{\pi X} - 4e^{\pi X} + 2\pi X + 5) +$$

$$+ \frac{1}{12}[e^{3\pi X} + 6e^{2\pi X}(1 - \pi X) + 3e^{\pi X}(2\pi^2 X^2 - 6\pi X + 5) - 6\pi X - 22] \tag{15}$$

Substituting eq. (7) into eq. (15), we obtain the approximate solution of eq. (6):

$$u(x)=\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}+e^{\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}}-\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}-1-$$

$$-\frac{1}{4}\left[-e^{2\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}}+4\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}e^{\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}}-4e^{\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}}+2\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}+5\right]+$$

$$+\frac{1}{12}\left(e^{3\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}}+6e^{2\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}}\left[1-\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right]+3e^{\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}}\left\{2\pi^{2}\left[\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right]^{2}-\right.\right.$$

$$\left.\left.-6\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}+5\right\}-6\pi\frac{x^{\alpha}}{\Gamma(\alpha+1)}-22\right) \tag{16}$$

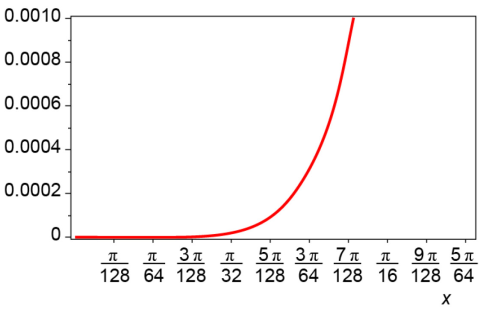

Figure 1. The absolute error at $\alpha = 1$ for eq. (6)

Remark 1. When $\alpha = 1$, the exact solution of eq. (6) is:

$$u(x) = -\ln[1 - \sin(\pi x)] \tag{17}$$

Remark 2. In tab. 1, we compute the values of the exact solution and the approximate solution for different values of α. By comparison, it is easy to find that the approximate solution continuously depend on the values of time-fractional derivative α. Figure 1, plots the absolute error between exact solutions and approximate solutions of eq. (6). It is clearly shown that proposed method is highly accurate.

Table 1. Comparison between the exact solution and the approximate solution of eq. (6) by ADM for different values of α

x	α			$U_{exa}(\alpha = 1)$
	0.4	0.6	1	
0.1	4.541994087	1.507134636	0.3696612036	0.3696400494
0.2	14.03193350	3.861634387	0.8887681117	0.8862108334
0.3	33.06009223	9.440343066	1.6944106310	1.6555708310
0.4	64.96052694	21.87003052	3.2220648200	3.2170890450

Conclusion

In this paper, we have successfully applied the fractional complex transform and Adomian decomposition method to find the solution of fractional Bratu-type equation. The numerical example shows that our proposed method is efficient and simple.

References

[1] He, J.-H., *et al.*, Variational Iteration Method for Bratu-Like Equation Arising in Electrospinning, *Carbohydrate Polymers, 105* (2013), May, pp. 229-230
[2] He, J.-H., Liu, H. M., Variational Approach to Nonlinear Problems and a Review on Mathematical Model of Electrospinning, *Nonlinear Analysis-Theory Methods & Applications, 63* (2005), 5-7, pp. 919-929
[3] Wan, Y. Q., *et al.*, Thermo-Electro-Hydrodynamic Model for Electrospinning Process, *Int. J. Nonlinear Sci. Numer., 5* (2004), 1, pp. 5-8
[4] Liu, H. Y., Wang, P., A Short Remark on WAN Model for Electrospinning and Bubble Electrospinning and Its Development, *Int. J. Nonlinear Sci. Numer., 16* (2015), 1, pp. 1-2
[5] Colantoni, A., Boubaker, K., Electro-Spun Organic Nanofibers Elaboration Process Investigations Using Comparative Analytical Solutions, *Carbohydrate Polymers, 101* (2014), Jan., pp. 307-312
[6] Podlubny, I., *Fractional Differential Equations*, Academic Press, New York, USA, 1999
[7] Hilfer, R., *Application of Fractional Calculus in Physics*, World Scientific, Singapore, 2000
[8] Hu, Y., He, J.-H., On Fractal Space-Time and Fractional Calculus, *Thermal Science, 20 (*2016), 3, pp. 773-777
[9] Wang, K. J., Pan, Z. L, An Analytical Model for Steady-State and Transient Temperature Fields in 3-D Integrated Circuits, *IEEE Trans. Compon., Packag., Manuf. Technol., 6* (2016), 7, pp. 1028-1041
[10] Wang, K. J., *et al.*, Integrated Microchannel Cooling in a Three Dimensional Integrated Circuit: A Thermal Management, *Thermal Science, 20* (2016), 3, pp. 899-902
[11] He, J.-H., Homotopy Perturbation Technique, *Computer Methods in Applied Mechanics and Engineering, 178* (1999), 3, pp. 257-262
[12] He, J.-H., A Coupling Method of a Homotopy Technique and a Perturbation Technique for Nonlinear Problems, *International Journal of Nonlinear Mechanics, 35* (2000), 1, pp. 37-43
[13] He, J.-H., Application of Homotopy Perturbation Method to Nonlinear Wave Equation, *Chaos, Solitons & Fractals, 26* (2005), 3, pp. 695-700
[14] Rajeev., Homotopy Perturbation Method for a Stefan Problem with Variable Latent Heat, *Thermal Science, 18* (2014), 2, pp. 391-398
[15] He, J.-H., A Short Remark on Fractional Variational Iteration Method, *Phys. Lett. A 375* (2011), 38, pp. 3362-3364
[16] He, J.-H., Variational Iteration Method – Some Recent Results and New Interpretations, *J. Comput. Appl. Math., 207* (2007), 1, pp. 3-17
[17] He, J.-H., Exp-Function Method for Fractional Differential Equations, *International Journal of Nonlinear Sciences and Numerical Simulations, 14* (2013), 6, pp. 363-366
[18] Ma, H. C., *et al.*, Exact Solutions of Nonlinear Fractional Partial Differential Equations by Fractional Sub-Equation Method, *Thermal Science, 19* (2015), 4, pp. 1239-1244
[19] Wazwaz, A. M., A Reliable Modification of Adomian Decomposition Method, *Appl. Math. Comput., 102* (1999), 1, pp. 77-86
[20] Adomian, G., A Review of the Decomposition Method in Applied Mathematics, *J. Math. Anal. Appl., 135* (1988), 2, pp. 501-504
[21] He, J.-H., Li, Z. B., Converting Fractional Differential Equations into Partial Differential Equations, *Thermal Science, 16* (2012), 2, pp. 331-334
[22] Li, Z., He, J.-H., Fractional Complex Transform for Fractional Differential Equations, *Math. Comput. Appl., 15* (2010), 5, pp. 970-973
[23] Wazwaz, A. M., Adomian Decomposition Method for a Reliable Treatment of the Bratu-Type Equations, *Appl. Math. Comput., 166* (2005), 3, pp. 652-663
[24] He, J.-H., *et al.*, A New Fractional Derivative and Its Application to Explanation of Polar Bear Hairs, *Journal of King Saud University Science*, *28* (2015), 2, pp. 190-192
[25] He, J.-H., A Tutorial Review on Fractal Spacetime and Fractional Calculus, *Int. J. Theor. Phys., 53* (2014), 11, pp. 3698-3718

Paper submitted: August 5, 2016
Paper revised: August 23, 2016
Paper accepted: October 25, 2016