NUMERICAL ANALYSIS OF THE (2+1)-DIMENSIONAL BOITI-LEON-PEMPINELLI EQUATION

by

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The (2+1)-dimensional Boiti-Leon-Pempinelli equation is studied by the modified variational iteration method. The numerical solutions to its initial value problem are provided and compared with the exact soliton solutions. The present theory offers an in-depth physical understanding of hydrodynamic properties of non-linear wave equations.

Key words: *Boiti-Leon-Pempinelli equation, soliton solution, modified variational iteration method*

Introduction

The investigation of exact solutions to non-linear evolution equations plays an important role in the non-linear physical areas. In this paper, we focus on the (2+1)-dimensional Boiti-Leon-Pempinelli (BLP) equation:

$$u_{ty} = (u^2 - u_x)_{xy} + 2v_{xxx}, \qquad v_t = v_{xx} + 2uv_x \tag{1}$$

The integrability of the BLP equation was studied in [1], by generalizing the sine-Gordon and sinh-Gordon equations. Equation (1) has been widely applied in the hydrodynamics for describing the evolution of the water waves [1, 2]. Various wave solutions of eq. (1) have been studied by means of the numerical or analytical methods, including improved projective Riccati equation method [3], generalized algebra method [4], tanh-coth method, exp-function method [5, 6], and Backlund transformation [7].

We are interested in the numerical behavior of the soliton solutions of the (2+1)-dimensional BLP equation. Precisely, we will give the numerical solutions to the initial value problem associated with the (2+1)-dimensional BLP equation, by applying the modified variational iteration method (MVIM) proposed in [8]. The approximate solutions by MVIM are compared with the exact soliton solutions. Numerical experiments show the efficiency of MVIM for the BLP equation.

Analysis of MVIM

In order to illustrate the basic idea of MVIM, let us consider the following partial differential equation:

$$Lu(x, y, t) + Ru(x, y, t) + Nu(x, y, t) = g(x, y, t), \qquad u(x, y, 0) = f(x, y) \tag{2}$$

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where $L = \partial/\partial t$, R is a linear operator with the partial derivative with respect to x and y, $Nu(x, y, t)$ – the non-linear term about $u(x, y, t)$, and $g(x, y, t)$ – the inhomogeneous term.

He [9, 10] proposed a variational iteration method (VIM) for solving eq. (2). The efficiency of VIM has been verified in different areas. For speeding up the convergence and reducing the computation cost of VIM, a MVIM was proposed in [8, 11]. The MVIM for eq. (2) is constructed by the following variational iteration formula:

$$u_{n+1}(x, y, t) = u_n(x, y, t) - \int_0^t \{R[u_n(x, y, \xi) - u_{n-1}(x, y, \xi)] + [G_n(x, y, \xi) - G_{n-1}(x, y, \xi)]\}\mathrm{d}\xi$$

with

$$u_{-1} = 0,\ u_0 = f(x, y),\ u_1 = u_0 - \int_0^t [R(u_0 - u_{-1}) + (G_0 - G_{-1}) - g]\mathrm{d}\xi$$

and

$G_n(x, y, t)$ is given by $Nu_n(x, y, t) = G_n(x, y, t) + O(t^{n+1})$

Numerical experiments

We consider the initial value problem of the (2+1)-dimensional BLP equation with the following initial conditions:

$$u(x, y, 0) = b + b\tanh(kx + py + l), \qquad v(x, y, 0) = c + p\tanh(kx + py + l)$$

where b, c, k, l, and p are arbitrary constants. The kink-shaped soliton solutions to eq. (1) are given by:

$$u(x, y, t) = b + b\tanh(kx + py - \lambda t + l), \quad \text{and} \quad v(x, y, t) = c + p\tanh(kx + py - \lambda t + l) \tag{3}$$

with $\lambda = 2b^2$[3].

By applying MVIM, the iteration formulae read:

$$u_{n+1}(x, y, t) = u_n(x, y, t) - \int_0^t \{[u_{nxxy}(x, y, \xi) - u_{n-1xxy}(x, y, \xi)] - 2[v_{nxxx}(x, y, \xi) - \\ - v_{n-1xxx}(x, y, \xi)] + [G_n(x, y, \xi) - G_{n-1}(x, y, \xi)]\}\mathrm{d}\xi \tag{4}$$

$$v_{n+1}(x, y, t) = v_n(x, y, t) + \int_0^t \{[v_{nxx}(x, y, \xi) - v_{n-1xx}(x, y, \xi)] + \\ + [H_n(x, y, \xi) - H_{n-1}(x, y, \xi)]\}\mathrm{d}\xi \tag{5}$$

where $G_n(x, y, t)$ is obtained from:

$$-(u_n^2)_{xy} = G_n(x, y, t) + O(t^{n+1})$$

and $H_n(x, y, t)$ is defined by:

$$2u_n v_{nx} = H_n(x, y, t) + O(t^{n+1})$$

where u_{-1}, v_{-1}, G_{-1}, and H_{-1} are set to zero.

By iteration eqs. (4) and (5) with the initial guesses $u_0 = u(x, y, 0)$ and $v_0 = v(x, y, 0)$, it follows the first order approximations:

$$u_1 = b + b\tanh\zeta - 2kpt\sec h^4(\zeta)[-2(b^2 + bk - 2k^2) + (b^2 + bk - 2k^2)\cosh(2\zeta) + b^2\sinh(2\zeta)],$$

$$v_1 = c + p\tanh\zeta + 2kpt\,\mathrm{sech}^2(\zeta)[b + (b-k)\tanh\zeta]$$

where $\zeta = kx + py + l$. The rest approximate solutions can be obtained similarly.

In this example, we first set $k = b = 0.1$, $p = l = 0.2$, and $c = 2$. Figure 1 plots the surfaces of the MVIM solutions u_3 and the kink-shaped soliton solutions $u(x, y, 0)$, see eq. (3), when $-10 \le x$, $y \le 10$, and $t = 0.5$. The surfaces for u_3 and $u(x, y, t)$ with $t = 1$ and $-10 \le x$, $y \le 10$ are presented in fig. 2. Similarly, figs. 3 and 4 plot the surfaces of the third order approximations v_3 and the kink-shaped soliton solutions $v(x, y, t)$, see eq. (3), with $t = 0.5$ and $t = 1$, respectively. For comparison, we present the surfaces of the relative errors of the approximate solutions u_3 and v_3 . Figures 5 and 6 show the error surfaces when with $t = 0.5$ and $t = 1$, respectively. In particular, the numerical results for the approximations u_3 and the exact soliton solutions $u(x, y, t)$ with $t = 0.5$ are shown in tab. 1. We also give the results for the approximate solutions v_3 when $t = 0.5$ in tab. 1. The errors of the MVIM solutions u_3 and v_3 are presented in tab. 2 when $t = 1$. It is easy to see that MVIM performs well for this example.

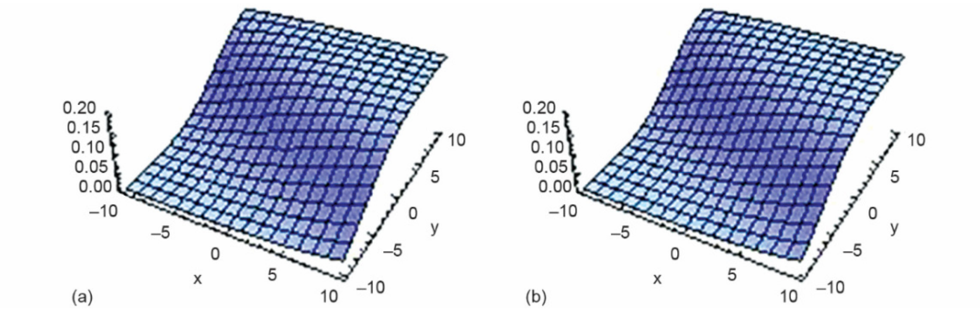

Figure 1. The MVIM solution u_3 (a) and the exact solution u (b) with $-10 \le x$, $y \le 10$ when $t = 0.5$

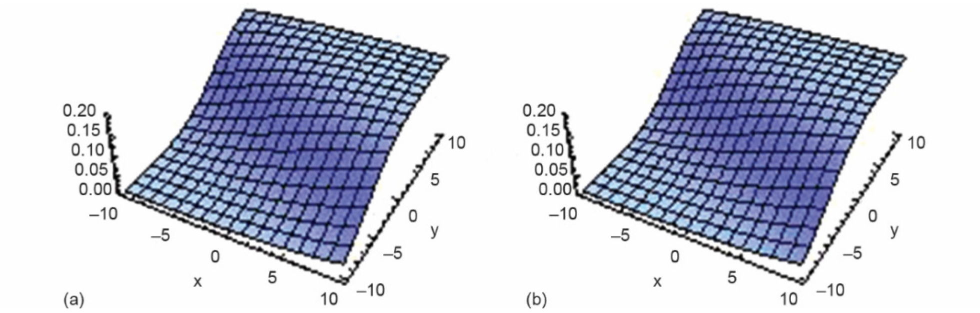

Figure 2. The approximation u_3 (a) and the exact solution u (b) with $-10 \le x, y \le 10$ when $t = 1$

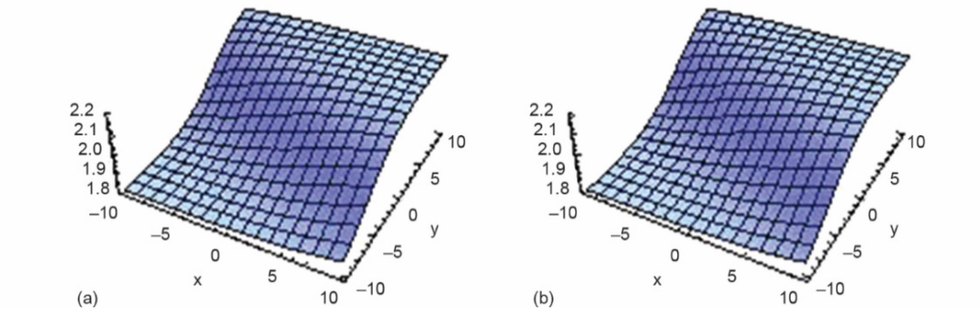

Figure 3. The MVIM solution v_3 (a) and the exact solution v (b) with $-10 \le x$, $y \le 10$ when $t = 0.5$

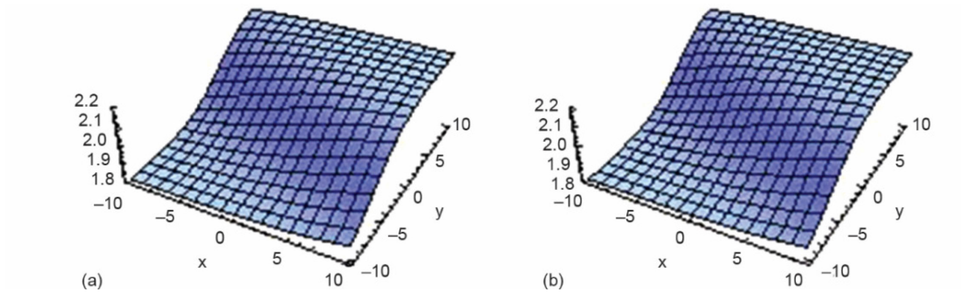

Figure 4. The approximation v_3 (a) and the exact solution v (b) with $-10 \le x$, y ≤ 10 when $t = 1$

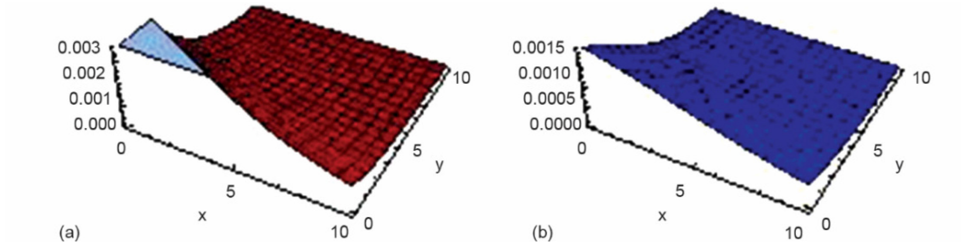

Figure 5. The error surfaces of MVIM solutions u_3 (a) and v_3 (b) with $t = 0.5$
(*for color image see journal web site*)

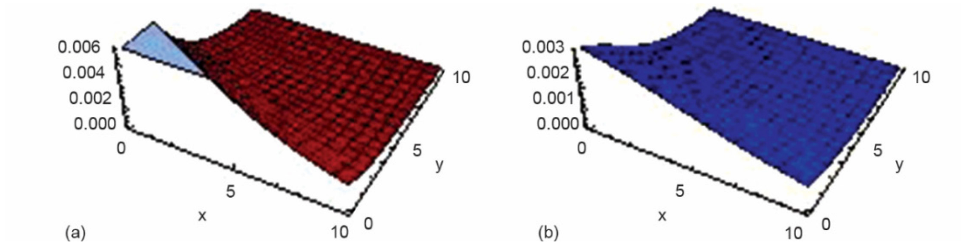

Figure 6. The error surfaces of MVIM solutions u_3 (a) and v_3 (b) with $t = 1$

Table 1. Relative errors of MVIM solutions u_3 and v_3 with $t = 0.5$

y	u_3				v_3			
	$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 1$	$x = 2$	$x = 3$	$x = 4$
1	$4.4\cdot10^{-3}$	$3.7\cdot10^{-3}$	$3.1\cdot10^{-3}$	$2.5\cdot10^{-3}$	$1.5\cdot10^{-3}$	$1.3\cdot10^{-3}$	$1.2\cdot10^{-3}$	$1.0\cdot10^{-3}$
2	$3.1\cdot10^{-3}$	$2.5\cdot10^{-3}$	$2.1\cdot10^{-3}$	$1.7\cdot10^{-3}$	$1.2\cdot10^{-3}$	$1.0\cdot10^{-3}$	$9.1\cdot10^{-4}$	$7.8\cdot10^{-4}$
3	$2.1\cdot10^{-3}$	$1.7\cdot10^{-3}$	$1.4\cdot10^{-3}$	$1.1\cdot10^{-3}$	$9.1\cdot10^{-4}$	$7.8\cdot10^{-4}$	$6.6\cdot10^{-4}$	$5.6\cdot10^{-4}$
4	$1.4\cdot10^{-3}$	$1.1\cdot10^{-3}$	$9.2\cdot10^{-4}$	$7.5\cdot10^{-4}$	$6.7\cdot10^{-4}$	$5.6\cdot10^{-4}$	$4.7\cdot10^{-4}$	$4.0\cdot10^{-4}$

We then consider the case with $k = p = 0.1$, $l = 0.5$, $b = 0.05$, and $c = 1$. Figures 7 and 8 show the compared results of the MVIM solutions u_3 and the kink-shaped soliton solutions $u(x, y, t)$ with $-10 \le x$, y ≤ 10 when $t = 0.5$ and $t = 1$, respectively. The surfaces of the approximations v_3 and the exact solutions $v(x, y, t)$ are presented in figs. 9 and 10. Figures 11 and 12 also plot the error surfaces of u_3 and v_3 when $t = 0.5$ and $t = 1$, respectively. As in the previous case, MVIM works well which offers an in-depth physical understanding of hydrodynamic properties of the BLP equation.

Table 2. Relative errors of MVIM solutions u_3 and v_3 with t = 1

y	u_3				v_3			
	x = 1	x = 2	x = 3	x = 4	x = 1	x = 2	x = 3	x = 4
1	$9.1\cdot10^{-3}$	$7.6\cdot10^{-3}$	$6.3\cdot10^{-3}$	$5.1\cdot10^{-3}$	$3.0\cdot10^{-3}$	$2.7\cdot10^{-3}$	$2.4\cdot10^{-3}$	$2.1\cdot10^{-3}$
2	$6.3\cdot10^{-3}$	$5.1\cdot10^{-3}$	$4.2\cdot10^{-3}$	$3.4\cdot10^{-3}$	$2.4\cdot10^{-3}$	$2.1\cdot10^{-3}$	$1.8\cdot10^{-3}$	$1.6\cdot10^{-3}$
3	$4.2\cdot10^{-3}$	$3.4\cdot10^{-3}$	$2.8\cdot10^{-3}$	$2.3\cdot10^{-3}$	$1.8\cdot10^{-3}$	$1.6\cdot10^{-3}$	$1.3\cdot10^{-3}$	$1.1\cdot10^{-3}$
4	$2.8\cdot10^{-3}$	$2.3\cdot10^{-3}$	$1.9\cdot10^{-3}$	$1.5\cdot10^{-3}$	$1.3\cdot10^{-3}$	$1.1\cdot10^{-3}$	$9.5\cdot10^{-4}$	$7.9\cdot10^{-4}$

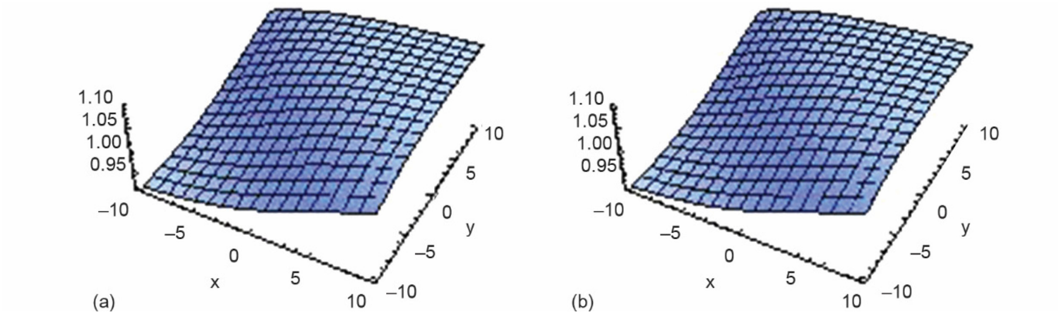

Figure 7. The numerical results for u_3 (a) and u (b) with –10 ≤ x, y ≤ 10 when t = 0.5

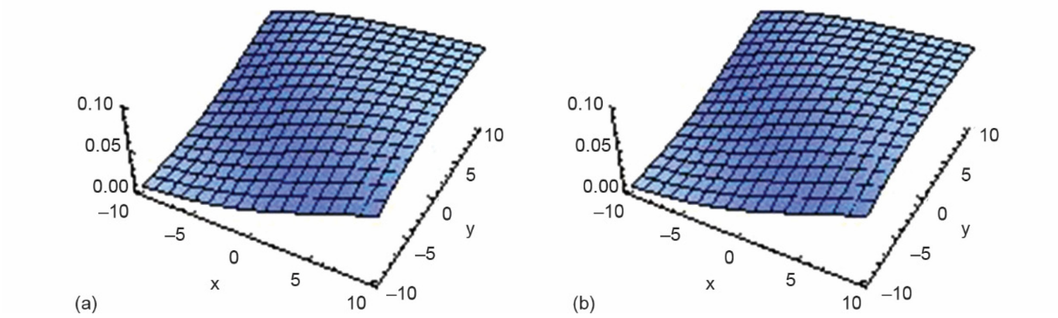

Figure 8. The numerical results for u_3 (a) and u (b) with –10 ≤ x, y ≤ 10 when t = 1

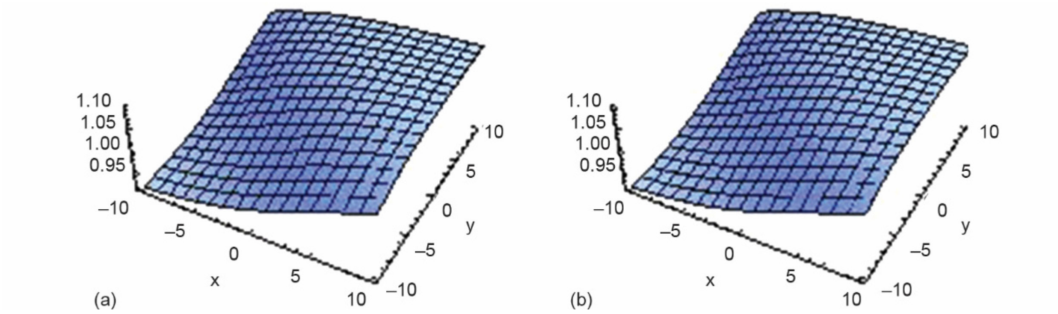

Figure 9. The compared results for v_3 (a) and v (b) with –10 ≤ x, y ≤ 10 when t = 0.5

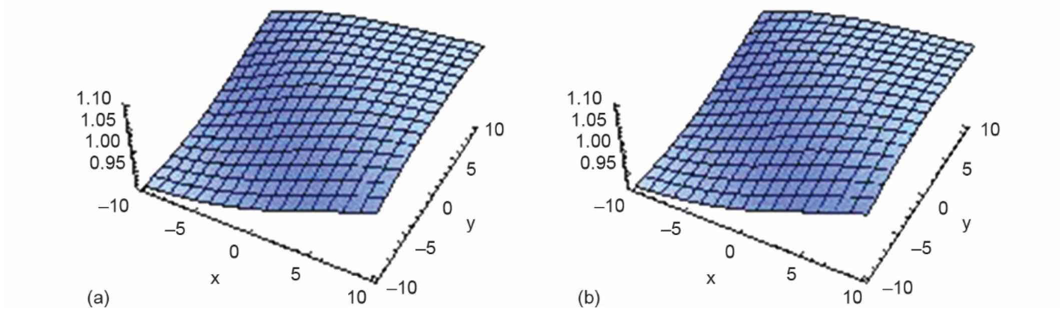

Figure 10. The compared results for v_3 (a) and v (b) with $-10 \le x, y \le 10$ when $t = 1$

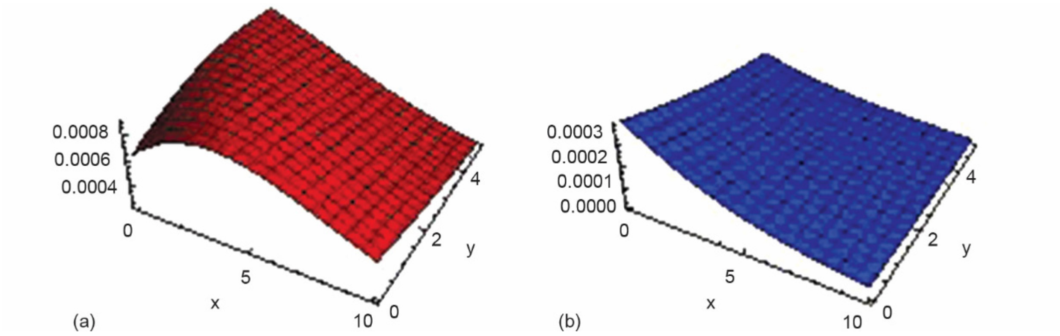

Figure 11. The error surfaces of MVIM solutions u_3 (a) and v_3 (b) with $t = 0.5$

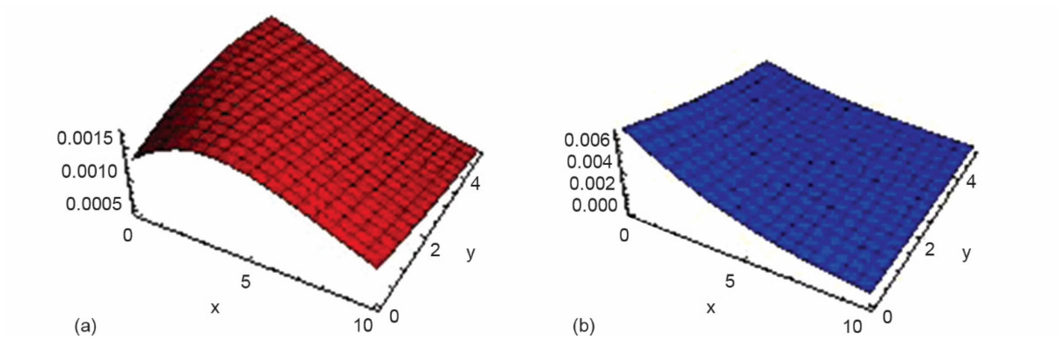

Figure 12. The error surfaces of MVIM solutions u_3 (a) and v_3 (b) with $t = 1$

Conclusion

This paper deals with the initial value problem of the (2+1)-dimensional BLP equation by using MVIM. The numerical results illustrate the efficiency of this method. We will further extend this method to other non-linear equations in our future work.

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