

INFINITE MANY CONSERVATION LAWS OF DISCRETE SYSTEM ASSOCIATED WITH A 3×3 MATRIX SPECTRAL PROBLEM

by

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Differential-difference equations are often considered as an alternative approach to describing some phenomena arising in heat/electron conduction and flow in carbon nanotubes and nanoporous materials. Infinite many conservation laws play important role in discussing the integrability of non-linear differential equations. In this paper, infinite many conservation laws of the non-linear differential-difference equations associated with a 3×3 matrix spectral problem are obtained.

Key words: conservation law, non-linear differential-difference equations, matrix spectral problem

Introduction

As is well known, searching for infinite many conservation laws is one of starting points to study the integrability of soliton equations [1]. Miura *et al.* [2] found a transformation, called Miura's transformation, between the Korteweg-de Vries (KdV) equation and the modified KdV (mKdV) equation. Taking advantage of Miura's transformation, one can transform any solution of the mKdV equation into the solution of the KdV equation, but not the other way [3]. More importantly, Miura's transformation proves that the KdV equation possesses infinite many conservation laws. Kruskal *et al.* [4] gave a guess, proved by Tu and Qin [5] later, on the number of conservation laws of a generalized KdV equation. Generally, most of non-linear differential equations which have soliton solutions exist infinite many conservation laws. Differential difference equation (DDE) play important role in modelling physical phenomena in mathematical physics, plasma physics, optical physics, biology, electric circuits, *etc.* [6], which are often considered as an alternative approach to describing some phenomena arising in heat/electron conduction and flow in carbon nanotubes and nanoporous materials, this is due to the fact that continuum hypothesis is no longer valid [7]. Based on such a reason, He and Zhu [8] earlier suggested some DDE to describe the fascinating phenomena arising in heat/electron conduction and flow in carbon nanotubes. In this paper, we shall obtain infinite many conservation laws of a new system of non-linear DDE [6]:

$$u_{n,t} = u_n (w_{n+1} - w_n) \quad (1)$$

$$v_{n,t} = 2u_n v_{n+1} + v_n w_n - v_{n-1} \quad (2)$$

$$w_{n,t} = u_{n-1} - u_n \quad (3)$$

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associated with a 3×3 matrix spectral problem

$$E\varphi_n = \begin{pmatrix} 0 & 0 & u_n \\ 0 & 1 & v_n \\ 1 & 0 & \lambda + w_n \end{pmatrix} \varphi_n \quad (4)$$

and the evolution equation

$$\varphi_{n,t} = \begin{pmatrix} \lambda + w_n & 0 & -u_{n-1} \\ 2v_n & -\lambda & -v_{n-1} \\ -1 & 0 & 0 \end{pmatrix} \varphi_n \quad (5)$$

where λ is a spectral parameter and $\lambda_t = 0$, $\varphi_n = (\varphi_{1n}, \varphi_{2n}, \varphi_{3n})^T$ – an eigenfunction vector, and T – the transpose of the vector or matrix, E – the shift operator defined by $E f(n, t) = f(n+1, t) \equiv f_{n+1}$, $E^{-1} f(n, t) = f(n-1, t) \equiv f_{n-1}$. It should be noted that recently Wen [6] obtained N -fold Darboux transformation and explicit solutions of eqs. (1)-(3). To the best of our knowledge, infinite many conservation laws of eqs. (1)-(3) have not been reported in the literature.

Infinite many conservation laws

It is easy to see from eq. (4) that:

$$\begin{pmatrix} E\varphi_{1n} \\ E\varphi_{2n} \\ E\varphi_{3n} \end{pmatrix} = \begin{pmatrix} 0 & 0 & u_n \\ 0 & 1 & v_n \\ 1 & 0 & \lambda + w_n \end{pmatrix} \begin{pmatrix} \varphi_{1n} \\ \varphi_{2n} \\ \varphi_{3n} \end{pmatrix} \quad (6)$$

namely

$$\varphi_{1,n+1} = u_n \varphi_{3,n}, \quad \varphi_{2,n+1} = \varphi_{2,n} + v_n \varphi_{3,n}, \quad \varphi_{3,n+1} = \varphi_{1,n} + (\lambda + w_n) \varphi_{3,n} \quad (7)$$

which shows

$$\varphi_{3,n+1} = u_{n-1} \varphi_{3,n-1} + (\lambda + w_n) \varphi_{3,n} \quad (8)$$

Supposing $\theta_n = \varphi_{3,n} / \varphi_{3,n+1}$ and substituting it into eq. (8) yields:

$$1 = u_{n-1} \frac{\varphi_{3,n-1}}{\varphi_{3,n}} \frac{\varphi_{3,n}}{\varphi_{3,n+1}} + (\lambda + w_n) \frac{\varphi_{3,n}}{\varphi_{3,n+1}} = u_{n-1} \theta_{n-1} \theta_n + (\lambda + w_n) \theta_n \quad (9)$$

On the other hand, from eq. (5) we have:

$$\begin{pmatrix} \varphi_{1n,t} \\ \varphi_{2n,t} \\ \varphi_{3n,t} \end{pmatrix} = \begin{pmatrix} \lambda + w_n & 0 & -u_{n-1} \\ 2v_n & -\lambda & -v_{n-1} \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_{1n} \\ \varphi_{2n} \\ \varphi_{3n} \end{pmatrix} \quad (10)$$

and hence obtains

$$\varphi_{1,n,t} = (\lambda + w_n) \varphi_{1,n} - u_{n-1} \varphi_{3,n}, \quad \varphi_{2,n,t} = 2v_n \varphi_{1,n} - \lambda \varphi_{2,n} - v_{n-1} \varphi_{3,n}, \quad \varphi_{3,n,t} = -\varphi_{1,n} \quad (11)$$

which gives

$$\varphi_{3,n,t} = -u_{n-1}\varphi_{3,n-1} \quad (12)$$

namely

$$\frac{\varphi_{3,n,t}}{\varphi_{3,n}} = -u_{n-1}\theta_{n-1} \quad (13)$$

Since:

$$-(\ln \theta_n)_t = \frac{\varphi_{1,n+1,t}}{\varphi_{1,n+1}} - \frac{\varphi_{1,n,t}}{\varphi_{1,n}} \quad (14)$$

the following relationship holds:

$$-(\ln \theta_n)_t = (E-1)(-u_{n-1}\theta_{n-1}) \quad (15)$$

If we set:

$$\theta_n = \sum_{j=1}^{\infty} \theta_n^{(j)} \lambda^{-j} \quad (16)$$

then eq. (9) becomes:

$$1 = u_{n-1} \left[\sum_{j=1}^{\infty} \theta_{n-1}^{(j)} \lambda^{-j} \right] \sum_{j=1}^{\infty} \theta_n^{(j)} \lambda^{-j} + (\lambda + w_n) \sum_{j=1}^{\infty} \theta_n^{(j)} \lambda^{-j} \quad (17)$$

Comparing the coefficients with same order of λ in eq. (17), we have:

$$\theta_n^{(1)} = 1, \quad \theta_n^{(2)} = -w_n, \quad \theta_n^{(3)} = w_n^2 - u_{n-1}, \quad \theta_n^{(4)} = 2w_n u_{n-1} + w_{n-1} u_{n-1} - w_n^3 \quad (18)$$

and so forth. At the same time, a recursion is derived:

$$\theta_n^{(m+1)} = - \left[u_{n-1} \sum_{k=1}^{m-1} \theta_{n-1}^{(k)} \theta_n^{(m-k)} + w_n \theta_n^{(m)} \right], \quad m \geq 2 \quad (19)$$

Substituting eq. (16) into eq. (15), we have:

$$- \left[\ln \sum_{j=1}^{\infty} \theta_n^{(j)} \lambda^{-j} \right]_t = (E-1) \left\{ -u_{n-1} \left[\sum_{j=1}^{\infty} \theta_{n-1}^{(j)} \lambda^{-j} \right] \right\} \quad (20)$$

and further gain:

$$\left\{ \sum_{k=1}^{\infty} (-1)^k \frac{1}{k} \left[\sum_{j=1}^{\infty} \theta_n^{(j+1)} \lambda^{-j} \right]^k \right\}_t = (E-1) \left\{ -u_{n-1} \left[\sum_{j=1}^{\infty} \theta_{n-1}^{(j)} \lambda^{-j} \right] \right\} \quad (21)$$

Comparing each of the coefficients with same order of λ in eq. (21), we finally obtain the following infinite many conservation laws of eqs. (1)-(3):

$$(w_n)_t = (E-1)(-u_{n-1}) \quad (22)$$

$$\left(u_{n-1} - \frac{1}{2}w_n^2\right)_t = (E-1)(u_{n-1}w_{n-1}) \quad (23)$$

$$\left(\frac{1}{3}w_n^3 - u_{n-1}w_{n-1} - u_{n-1}w_n\right)_t = (E-1)\left[-u_{n-1}(w_{n-1}^2 - u_{n-2})\right] \quad (24)$$

$$\begin{aligned} &\left(-u_{n-2}u_{n-1} - \frac{1}{2}u_{n-1}^2 + u_{n-1}w_{n-1}^2 + u_{n-1}w_{n-1}w_n + u_{n-1}w_n^2 - \frac{1}{4}w_n^4\right)_t = \\ &= (E-1)\left[-u_{n-1}(u_{n-2}w_{n-2} + 2u_{n-2}w_{n-1} - w_{n-1}^3)\right] \end{aligned} \quad (25)$$

$$\begin{aligned} &\left(u_{n-2}u_{n-1}w_{n-2} + 2u_{n-2}u_{n-1}w_{n-1} + u_{n-1}^2w_{n-1} - u_{n-1}w_{n-1}^3 + u_{n-2}u_{n-1}w_n + u_{n-1}^2w_n - u_{n-1}w_{n-1}^2w_n - \right. \\ &\left. - u_{n-1}w_{n-1}w_n^2 - u_{n-1}w_n^3 + \frac{1}{5}w_n^5\right)_t = (E-1)\left[-u_{n-1}(u_{n-3}u_{n-2} + u_{n-2}^2 - u_{n-2}w_{n-2}^2 - \right. \\ &\left. - 2u_{n-2}w_{n-2}w_{n-1} - 3u_{n-2}w_{n-1}^2 + w_{n-1}^4)\right] \end{aligned} \quad (26)$$

$$\begin{aligned} &\left(u_{n-3}u_{n-2}u_{n-1} + u_{n-2}^2u_{n-1} + u_{n-2}u_{n-1}^2 + \frac{1}{3}u_{n-1}^3 - u_{n-2}u_{n-1}w_{n-2}^2 - 2u_{n-2}u_{n-1}w_{n-2}w_{n-1} - 3u_{n-2}u_{n-1}w_{n-1}^2 - \right. \\ &\left. - \frac{3}{2}u_{n-1}^2w_{n-1}^2 + u_{n-1}w_{n-1}^4 - u_{n-2}u_{n-1}w_{n-2}w_n - 2u_{n-2}u_{n-1}w_{n-1}w_n - 2u_{n-1}^2w_{n-1}w_n + \right. \\ &\left. + u_{n-1}w_{n-1}^3w_n - u_{n-2}u_{n-1}w_n^2 - \frac{3}{2}u_{n-1}^2w_n^2 + u_{n-1}w_{n-1}^2w_n^2 + u_{n-1}w_{n-1}w_n^3 + u_{n-1}w_n^4 - \frac{w_n^6}{6}\right)_t = \\ &= (E-1)\left[-u_{n-1}(-u_{n-3}u_{n-2}w_{n-3} - 2u_{n-3}u_{n-2}w_{n-2} - 2u_{n-2}^2w_{n-2} + u_{n-2}w_{n-2}^3 - \right. \\ &\left. - 2u_{n-3}u_{n-2}w_{n-1} - 3u_{n-2}^2w_{n-1} + 2u_{n-2}w_{n-2}^2w_{n-1} + 3u_{n-2}w_{n-2}w_{n-1}^2 + 4u_{n-2}w_{n-1}^3 - w_{n-1}^5)\right] \end{aligned} \quad (27)$$

$$\begin{aligned} &\left(-u_{n-3}u_{n-2}u_{n-1}w_{n-3} - 2u_{n-3}u_{n-2}u_{n-1}w_{n-2} - 2u_{n-2}^2u_{n-1}w_{n-2} - u_{n-2}u_{n-1}^2w_{n-2} + u_{n-2}u_{n-1}w_{n-2}^3 - \right. \\ &\left. - 2u_{n-3}u_{n-2}u_{n-1}w_{n-1} - 3u_{n-2}^2u_{n-1}w_{n-1} - 3u_{n-2}u_{n-1}^2w_{n-1} - u_{n-1}^3w_{n-1} + 2u_{n-2}u_{n-1}w_{n-2}^2w_{n-1} + \right. \\ &\left. + 3u_{n-2}u_{n-1}w_{n-2}w_{n-1}^2 + 4u_{n-2}u_{n-1}w_{n-1}^3 + 2u_{n-1}^2w_{n-1}^3 - u_{n-1}w_{n-1}^5 - u_{n-3}u_{n-2}u_{n-1}w_n - \right. \\ &\left. - u_{n-2}^2u_{n-1}w_n - 2u_{n-2}u_{n-1}^2w_n - u_{n-1}^3w_n + u_{n-2}u_{n-1}w_{n-2}^2w_n + 2u_{n-2}u_{n-1}w_{n-2}w_{n-1}w_n + \right. \\ &\left. + 3u_{n-2}u_{n-1}w_{n-1}^2w_n + 3u_{n-1}^2w_{n-1}^2w_n - u_{n-1}w_{n-1}^4w_n + 2u_{n-2}u_{n-1}w_{n-1}w_n^2 + 3u_{n-1}^2w_{n-1}w_n^2 - \right. \end{aligned}$$

$$\begin{aligned}
 & -u_{n-1}w_{n-1}^2w_n^2 + u_{n-2}u_{n-1}w_n^3 + 2u_{n-1}^2w_n^3 - u_{n-1}w_{n-1}^2w_n^3 - u_{n-1}w_{n-1}^2w_n^3 - u_{n-1}w_{n-1}w_n^4 - \\
 & \left. -u_{n-1}w_n^5 + \frac{w_n^7}{7} \right)_t = (E-1) \left[-u_{n-1}(-u_{n-4}u_{n-3}u_{n-2} - u_{n-3}^2u_{n-2} - 2u_{n-3}u_{n-2}^2 - u_{n-2}^3 + \right. \\
 & + u_{n-3}u_{n-2}w_{n-3}^2 + 2u_{n-3}u_{n-2}w_{n-3}w_{n-2} + 3u_{n-3}u_{n-2}w_{n-2}^2 + 3u_{n-2}^2w_{n-2}^2 - u_{n-2}w_{n-2}^4 + \\
 & + 2u_{n-3}u_{n-2}w_{n-3}w_{n-1} - 2u_{n-2}w_{n-2}^3w_{n-1} + 3u_{n-3}u_{n-2}w_{n-1}^2w_{n-2}w_{n-1} + 6u_{n-2}^2w_{n-1}^2 - \\
 & \left. -3u_{n-2}w_{n-2}^2w_{n-1}^2 - 4u_{n-2}w_{n-2}w_{n-1}^3 - 5u_{n-2}w_{n-1}^4 + w_{n-1}^6) \right] \quad (28) \\
 & \left(-u_{n-4}u_{n-3}u_{n-2}u_{n-1} - u_{n-3}^2u_{n-2}u_{n-1} - 2u_{n-3}u_{n-2}^2u_{n-1} - u_{n-2}^3u_{n-1} - u_{n-3}u_{n-2}u_{n-1}^3 - \frac{3}{2}u_{n-1}^2u_{n-2}^2 - \right. \\
 & -u_{n-2}u_{n-1}^3 - \frac{1}{4}u_{n-1}^4 + u_{n-3}u_{n-2}u_{n-1}w_{n-3}^2 + 2u_{n-3}u_{n-2}u_{n-1}w_{n-3}w_{n-2} + 3u_{n-3}u_{n-2}u_{n-1}w_{n-2}^2 + \\
 & + 3u_{n-2}^2u_{n-1}w_{n-2}^2 + u_{n-2}u_{n-1}^2w_{n-2}^2 - u_{n-2}u_{n-1}w_{n-2}^4 + 2u_{n-3}u_{n-2}u_{n-1}w_{n-3}w_{n-1} + \\
 & + 4u_{n-3}u_{n-2}u_{n-1}w_{n-2}w_{n-1} + 6u_{n-2}^2u_{n-2}w_{n-1} + 3u_{n-2}u_{n-1}^2w_{n-2}w_{n-1}w_n + \\
 & + u_{n-2}u_{n-1}w_{n-2}^2w_n + 2u_{n-2}u_{n-1}w_{n-2}w_{n-1}w_n + 3u_{n-2}u_{n-1}w_{n-1}^2w_n + 3u_{n-1}^2w_{n-1}^2w_n - u_{n-1}w_{n-1}^4w_n + \\
 & + 2u_{n-2}u_{n-1}w_{n-1}w_n^2 - 2u_{n-2}u_{n-1}w_{n-2}^3w_{n-1} + 3u_{n-3}u_{n-2}u_{n-1}w_{n-1}^2 + 6u_{n-2}^2u_{n-1}w_{n-1}^2 + \\
 & + 6u_{n-2}u_{n-1}^2w_{n-1}^2 + 2u_{n-1}^3w_{n-1}^2 - 3u_{n-2}u_{n-1}w_{n-2}^2w_{n-1}^2 - 4u_{n-2}u_{n-1}w_{n-2}w_{n-1}^3 - 5u_{n-2}u_{n-1}w_{n-1}^4 - \\
 & -\frac{5}{2}u_{n-1}^2w_{n-1}^4 + u_{n-1}w_{n-1}^6 + u_{n-3}u_{n-2}w_{n-1}w_{n-3}w_n + 2u_{n-3}u_{n-2}u_{n-1}w_{n-2}w_n + 2u_{n-2}^2u_{n-1}w_{n-2}w_n + \\
 & + 2u_{n-2}u_{n-1}^2w_{n-2}w_n - u_{n-2}u_{n-1}w_{n-1}^3w_n + 2u_{n-3}u_{n-2}u_{n-1}w_{n-1}w_n + 3u_{n-2}^2u_{n-1}w_{n-1}w_n + \\
 & + 6u_{n-2}u_{n-1}^2w_{n-1}w_n + 3u_{n-1}^3w_{n-1}w_n - 2u_{n-2}u_{n-1}w_{n-2}^2w_{n-1}w_n - 3u_{n-2}u_{n-1}w_{n-2}w_{n-1}^2w_n - \\
 & -4u_{n-2}u_{n-1}w_{n-1}^3w_n - 4u_{n-1}^2w_{n-1}^3w_n + u_{n-1}w_{n-1}^5w_n + u_{n-3}u_{n-2}u_{n-1}w_n^2 + u_{n-2}^2u_{n-1}w_n^2 + \\
 & + 3u_{n-2}u_{n-1}^2w_n^2 + 2u_{n-1}^3w_n^2 - u_{n-2}u_{n-1}w_{n-2}^2w_n^2 - 2u_{n-2}u_{n-1}w_{n-2}w_{n-1}w_n^2 - \\
 & -3u_{n-2}u_{n-1}w_{n-1}^2w_n^2 - \frac{9}{2}u_{n-1}^2w_{n-1}^2w_n^2 + u_{n-1}w_{n-1}^4w_n^2 - u_{n-2}u_{n-1}w_{n-2}w_n^3 - 2u_{n-2}u_{n-1}w_{n-1}w_n^3 - \\
 & -4u_{n-1}^2w_{n-1}w_n^3 + u_{n-1}w_{n-1}^3w_n^3 - u_{n-2}u_{n-1}w_n^4 - \frac{5}{2}u_{n-1}^2w_n^4 + u_{n-1}w_{n-1}^2w_n^4 + \\
 & \left. + u_{n-1}w_{n-1}w_n^5 + u_{n-1}w_n^6 - \frac{w_n^8}{8} \right)_t = (E-1) \left[-u_{n-1}(-u_{n-4}u_{n-3}u_{n-2}w_{n-4} + 2u_{n-4}u_{n-3}u_{n-2}w_{n-3} + \right.
 \end{aligned}$$

$$\begin{aligned}
 &+2u_{n-3}^2u_{n-2}w_{n-3} + 2u_{n-3}u_{n-2}^2w_{n-3} - u_{n-3}u_{n-2}w_{n-3}^3 + 2u_{n-4}u_{n-3}u_{n-2}w_{n-2} + 3u_{n-3}^2u_{n-2}w_{n-2} + \\
 &\quad + 6u_{n-3}u_{n-2}^2w_{n-2} + 3u_{n-2}^3w_{n-2} - 2u_{n-3}u_{n-2}u_{n-3}^2w_{n-2} - 3u_{n-3}u_{n-2}w_{n-3}w_{n-2}^2 - \\
 &-4u_{n-3}u_{n-2}w_{n-2}^3 - 4u_{n-2}^2w_{n-2}^3 + u_{n-2}w_{n-2}^5 + 6u_{n-3}u_{n-2}^2w_{n-1} + 4u_{n-2}^3w_{n-1} - 2u_{n-3}w_{n-3}^2w_{n-1} - \\
 &\quad -4u_{n-3}u_{n-2}w_{n-3}w_{n-2}w_{n-1} - 6u_{n-3}u_{n-2}w_{n-2}^2w_{n-1} - 9u_{n-2}^2w_{n-2}^2w_{n-1} + 2u_{n-2}w_{n-2}^4w_{n-1} - \\
 &\quad -3u_{n-3}u_{n-2}w_{n-3}w_{n-1}^2 - 6u_{n-3}u_{n-2}w_{n-2}w_{n-1}^2 - 12u_{n-2}^2w_{n-2}w_{n-1}^2 + 3u_{n-2}w_{n-2}^3w_{n-1}^2 - \\
 &\quad -4u_{n-3}u_{n-2}w_{n-1}^2 - 10u_{n-2}^2w_{n-1}^3 + 4u_{n-2}w_{n-2}^2w_{n-1}^3 + 5u_{n-2}w_{n-2}w_{n-1}^4 + 6u_{n-2}w_{n-1}^5 - w_{n-1}^7) \Big] \quad (29)
 \end{aligned}$$

and so on.

Conclusion

We have obtained infinite many conservation laws of the non-linear DDE (1)-(3) associated with a 3×3 matrix spectral problem (4) and its evolution eq. (5). Recently, fractional-order differential calculus and its applications and hierarchies of non-linear partial differential equations (PDE) with variable coefficients have attached much attention, such as those in [9-20]. Constructing conservation laws of fractional-order non-linear differential equations and variable-coefficient hierarchies of non-linear PDE is worthy of study. This is our task in future.

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