ANALYTICAL TREATMENT ON A NEW GENERALIZED ABLOWITZ-KAUP-NEWELL-SEGUR HIERARCHY OF THERMAL AND FLUID EQUATIONS

by

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Constructing analytical solutions for non-liner partial differential equations arising in thermal and fluid science is important and interesting. In this paper, Hirota's bi-linear method is extended to a new generalized Ablowitz-Kaup-Newell-Segur hierarchy which includes heat conduction equation, advection equation, advection-dispersion equation, and Korteweg-de Vries equation as special cases. As a result, bi-linear form of the generalized Ablowitz-Kaup-Newell-Segur hierarchy is derived. Based on the derived bi-linear form, exact and explicit n-soliton solutions of the generalized Ablowitz-Kaup-Newell-Segur hierarchy are obtained.

Key words: *analytical solution, Hirota's bi-linear method, bi-linear form, generalized Ablowitz-Kaup-Newell-Segur hierarchy, n-soliton solution*

Introduction

Recently, constructing analytical solutions of non-linear PDE is relatively active because of searching for as many as meaningful exact solutions are of both theoretical and practical value. In the past several decades, mathematicians and physicists have made many significant researches in this direction and some effective methods have been proposed [1-12]. It was shown that direct algorithm [11] of the exp-function method [4] for non-linear PDE has an advantage in dealing with the so-called middle expression expansion problem. Among these existing methods, Hirota's bi-linear method [2] is a direct method for constructing multi-soliton solutions of non-linear PDE. However, there is little research work in extending Hirota's bi-linear method to a whole hierarchy of non-linear PDE. It is because that transforming a given hierarchy of non-linear PDE to its bi-linear form is very difficult. In this paper, we shall extend Hirota's bi-linear method [2] to the generalized Ablowitz-Kaup-Newell-Segur (gAKNS) hierarchy with time-dependent coefficients:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \sum_{i=0}^{m} a_i(t) L^i \begin{pmatrix} -q \\ r \end{pmatrix}, \qquad (m = 1, 2, ...) \tag{1}$$

where $a_i(t)$ are the arbitrary continuous functions of t, the operator L is employed:

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$$L=\sigma\partial+2\begin{pmatrix} q\\ -r\end{pmatrix}\partial^{-1}(r,q)\,,\quad \partial=\frac{\partial}{\partial x},\quad \partial^{-1}=\frac{1}{2}\int_{-\infty}^{x}\mathrm{d}x-\frac{1}{2}\int_{x}^{+\infty}\mathrm{d}x\,,\quad \sigma=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \tag{2}$$

It should be noted that eq. (1) generalizes the known AKNS hierarchy [12]:

$$\begin{pmatrix} q\\ r\end{pmatrix}_t=L^n\begin{pmatrix} -q\\ r\end{pmatrix},\quad (n=0,\,1,\,2,\ldots) \tag{3}$$

For a special case of eq. (1), we take $m = 3$ and gain the gAKNS equations with time-dependent coefficients [13]:

$$q_t=a_3(t)(q_{xxx}-6qrq_x)+a_2(t)(-q_{xx}+2q^2r)+a_1(t)q_x-a_0(t)q \tag{4}$$

$$r_t=a_3(t)(r_{xxx}-6qrr_x)+a_2(t)(r_{xx}-2r^2q)+a_1(t)r_x+a_0(t)r \tag{5}$$

which include the heat conduction equation $q_t = q_{xx}$, the advection equation $q_t = q_x$, the advection-dispersion equation $q_t + q_x = q_{xx}$, and the KdV equation – a mathematical model of waves on shallow water surfaces $q_t = 6qq_x + q_{xx}$ as some special cases.

Main results

Theorem 1. Taking the following rational transformation:

$$q=\frac{g}{f},\quad r=\frac{h}{f},\quad f=g(x,\,t)\,,\quad g=g(x,t)\,,\quad h=h(x,\,t) \tag{6}$$

the gAKNS hierarchy (1) can be bi-linearized:

$$\left[\mathrm{D}_t+\sum_{i=0}^{m}(-1)^i a_i(t)\mathrm{D}_x^i\right]g\cdot f=0\,,\qquad \left[\mathrm{D}_t-\sum_{i=0}^{m}a_i(t)\mathrm{D}_x^i\right]h\cdot f=0 \tag{7*}$$

under the condition that:

$$\mathrm{D}_x^2 f\cdot f+2gh=0 \tag{8}$$

where D_x and D_t are Hirota's differential operators [7], and $\mathrm{D}_x^0 f\cdot g=\mathrm{D}_t^0 f\cdot g=fg$ is assumed.

Theorem 2. The gAKNS hierarchy (1) possesses exact and explicit n-soliton solutions determined by eq. (6) with:

$$f=\sum_{\mu=0,1}A_1(\mu)\mathrm{e}^{\sum_{i=1}^{2n}\mu_i\xi_i+\sum_{1\le i<j}^{2n}\mu_i\mu_j\theta_{ij}},\qquad g=\sum_{\mu=0,1}A_2(\mu)\mathrm{e}^{\sum_{i=1}^{2n}\mu_i\xi_i+\sum_{1\le i<j}^{2n}\mu_i\mu_j\theta_{ij}},$$
$$h=\sum_{\mu=0,1}A_3(\mu)\mathrm{e}^{\sum_{i=1}^{2n}\mu_i\xi_i+\sum_{1\le i<j}^{2n}\mu_i\mu_j\theta_{ij}} \tag{9}$$

$$\xi_i=k_ix-\sum_{j=0}^{m}(-1)^j k_i^j\int_0^t a_j(s)\mathrm{d}s+\xi_i^0,\qquad \eta_i=l_ix+\sum_{j=0}^{m}l_i^j\int_0^t a_j(s)\mathrm{d}s+\eta_i^0 \tag{10}$$

$$\mathrm{e}^{\theta_{ij}}=-(k_i-k_j)^2,\qquad \mathrm{e}^{\theta_{(i+n)(j+n)}}=-(l_i-l_j)^2\,,\quad (i<j=2,3,\cdots,n) \tag{11}$$

* Remark: all the symbols · in this paper denote a computation

$$\mathrm{e}^{\theta_{i(j+n)}} = -\frac{1}{(k_i + l_j)^2}, \quad (i,\ j = 1,2,\cdots,n) \tag{12}$$

where ξ_i^0 and η_i^0 are arbitrary constants, the summation $\sum_{\mu=0,1}$ refers to all possible combinations of each $\mu_i = 0,1$ for $i = 1, 2, \ldots, n$, $A_1(\mu)$, $A_2(\mu)$, and $A_3(\mu)$ denote that when selecting all the possible combinations $\mu_i = 0,1$ ($j = 1, 2,\ldots,2n$) the following conditions hold, respectively:

$$\sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \mu_{n+j}, \quad \sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \mu_{n+j} + 1, \quad \sum_{j=1}^{n} \mu_j + 1 = \sum_{j=1}^{n} \mu_{n+j} \tag{13}$$

Proofs of *theorems*

Theorem 1. For conveniently, we first suppose that:

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_i} = L^i \begin{pmatrix} -q \\ r \end{pmatrix}, \quad \begin{pmatrix} q \\ r \end{pmatrix}_{t_{i+1}} = L^{i+1} \begin{pmatrix} -q \\ r \end{pmatrix} = L \begin{pmatrix} q \\ r \end{pmatrix}_{t_i} \tag{14}$$

namely

$$q_{t_{i+1}} = -q_{t_i,x} + 2q\partial^{-1}(qr)_{t_i}, \quad r_{t_{i+1}} = r_{t_i,x} - 2r\partial^{-1}(qr)_{t_i} \tag{15}$$

Substituting eq. (6) into eq. (15) and using eq. (8) we have:

$$\mathrm{D}_{t_{i+1}} g \cdot f = -\mathrm{D}_{t_i} \mathrm{D}_x g \cdot f, \qquad \mathrm{D}_{t_{i+1}} h \cdot f = \mathrm{D}_{t_i} \mathrm{D}_x h \cdot f \tag{16}$$

We next rewrite the right-hand side of eq. (1):

$$\sum_{i=0}^{m} a_i(t) L^i \begin{pmatrix} -q \\ r \end{pmatrix} = a_0(t) \begin{pmatrix} -q \\ r \end{pmatrix} + \sum_{i=1}^{m} a_i(t) \begin{pmatrix} q \\ r \end{pmatrix}_{t_{i-1}} \tag{17}$$

In view of eqs. (6) and (17), we transform the right-hand side of eq. (1):

$$\frac{1}{f^2}\left[a_0(t)\begin{pmatrix} -gf \\ hf \end{pmatrix} + \sum_{i=1}^{m} a_i(t) \begin{pmatrix} \mathrm{D}_{t_i} g \cdot f \\ \mathrm{D}_{t_i} h \cdot f \end{pmatrix}\right] = \frac{1}{f^2}\left\{\sum_{i=0}^{m} a_i(t) \begin{bmatrix} (-1)^{i+1} \mathrm{D}_x^i g \cdot f \\ \mathrm{D}_x^i h \cdot f \end{bmatrix}\right\} \tag{18}$$

On the other hand, with the substitution of eq. (6) the left-hand side of eq. (1) becomes:

$$\frac{1}{f^2}\begin{pmatrix} g_t f - g f_t \\ h_t f - h f_t \end{pmatrix} = \frac{1}{f^2}\begin{pmatrix} \mathrm{D}_t g \cdot f \\ \mathrm{D}_t h \cdot f \end{pmatrix} \tag{19}$$

Then we easily derive eq. (7) from eqs. (18) and (19). The proof of *Theorem 1* is ended.

Theorem 2. We expand *g, h,* and *f*:

$$g = \sum_{i=1}^{n} \varepsilon^{2i-1} g_{2i-1}, \quad h = \sum_{i=1}^{n} \varepsilon^{2i-1} h_{2i-1}, \quad f = \sum_{i=0}^{n} \varepsilon^{2i} f_{2i} (f_0 = 1) \tag{20}$$

Substituting eq. (20) into eqs. (7) and (8) and then collecting all the coefficients with same order of ε, we get a system of differential equations (SDE):

$$\varepsilon: \quad g_{1t} + \sum_{i=0}^{m} (-1)^i a_i(t)\partial_x^i g_1 = 0, \qquad h_{1t} - \sum_{i=0}^{m} (-1)^i a_i(t)\partial_x^i h_1 = 0 \tag{21}$$

$$\varepsilon^2: \quad f_{2xx} = -g_1 h_1 \tag{22}$$

$$\varepsilon^3: \quad \left[\mathrm{D}_t + \sum_{i=0}^{m}(-1)^i a_i(t)\mathrm{D}_x^i\right](g_3 \cdot 1 + g_1 \cdot f_2) = 0, \quad \left[\mathrm{D}_t - \sum_{i=0}^{m} a_i(t)\mathrm{D}_x^i\right](h_3 \cdot 1 + h_1 \cdot f_2) = 0 \tag{23}$$

$$\varepsilon^4: \quad f_{4xx} = -g_2 h_2 \tag{24}$$

and so forth.

It is easy to see that eq. (21) has solutions:

$$g_1 = \mathrm{e}^{\xi_1}, \quad \xi_1 = k_1 x - \sum_{i=0}^{m} (-1)^i k_1^i \int_0^t a_i(s)\mathrm{d}s + \xi_1^0 \tag{25}$$

$$h_1 = \mathrm{e}^{\eta_1}, \quad \eta_1 = l_1 x + \sum_{i=0}^{m} l_1^i \int_0^t a_i(s)\mathrm{d}s + \eta_1^0 \tag{26}$$

Substituting eqs. (25) and (26) into eq. (22), yields:

$$f_2 = \mathrm{e}^{\xi_1 + \eta_1 + \theta_{13}}, \quad \mathrm{e}^{\theta_{13}} = -\frac{1}{(k_1 + l_1)^2} \tag{27}$$

If $g_3 = h_3 = f_4 = ... = 0$, then eqs. (25)-(27) satisfy all the other equations in the previous SDE. We therefore obtain one-soliton solutions of eq. (1):

$$q = \frac{\mathrm{e}^{\xi_1}}{1 + \mathrm{e}^{\xi_1 + \eta_1 + \theta_{13}}}, \quad r = \frac{\mathrm{e}^{\eta_1}}{1 + \mathrm{e}^{\xi_1 + \eta_1 + \theta_{13}}} \tag{28}$$

To construct two-soliton solutions, we select another pair of solutions of eq. (21):

$$g_1 = \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2}, \quad \xi_i = k_i x - \sum_{j=0}^{m} (-1)^j k_i^j \int_0^t a_j(s)\mathrm{d}s + \xi_i^0, \quad (i = 1, 2) \tag{29}$$

$$h_1 = \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2}, \quad \eta_i = l_i x + \sum_{j=0}^{3} l_i^j \int a_j(t)\mathrm{d}t + \eta_i^0, \quad (i = 1, 2) \tag{30}$$

then solving eqs. (22)-(24) yields:

$$f_2 = \mathrm{e}^{\xi_1 + \eta_1 + \theta_{13}} + \mathrm{e}^{\xi_1 + \eta_2 + \theta_{14}} + \mathrm{e}^{\xi_2 + \eta_1 + \theta_{23}} + \mathrm{e}^{\xi_2 + \eta_2 + \theta_{24}} \tag{31}$$

$$g_3 = \mathrm{e}^{\xi_1 + \xi_2 + \eta_1 + \theta_{12} + \theta_{13} + \theta_{23}} + \mathrm{e}^{\xi_1 + \xi_2 + \eta_2 + \theta_{12} + \theta_{14} + \theta_{24}} \tag{32}$$

$$h_3 = \mathrm{e}^{\xi_1 + \eta_1 + \eta_2 + \theta_{13} + \theta_{14} + \theta_{34}} + \mathrm{e}^{\xi_2 + \eta_1 + \eta_2 + \theta_{23} + \theta_{24} + \theta_{34}} \tag{33}$$

$$f_4 = \mathrm{e}^{\xi_1 + \xi_2 + \eta_1 + \eta_2 + \theta_{12} + \theta_{13} + \theta_{14} + \theta_{23} + \theta_{24} + \theta_{34}} \tag{34}$$

where

$$\mathrm{e}^{\theta_{12}} = -(k_1 - k_2)^2, \quad \mathrm{e}^{\theta_{34}} = -(l_1 - l_2)^2, \quad \mathrm{e}^{\theta_{i(j+2)}} = -\frac{1}{(k_i + l_j)^2}, \quad (i,\ j = 1,\ 2) \tag{35}$$

We can verify that if $g_5 = h_5 = f_6 = ... = 0$, then those unwritten equations in the previous SDE all hold. Thus, we obtain two-soliton solutions of eq. (1):

$$q = \frac{\mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} + \mathrm{e}^{\xi_1+\xi_2+\eta_1+\theta_{12}+\theta_{13}+\theta_{23}} + \mathrm{e}^{\xi_1+\xi_2+\eta_2+\theta_{12}+\theta_{14}+\theta_{24}}}{1+\mathrm{e}^{\xi_1+\eta_1+\theta_{13}} + \mathrm{e}^{\xi_1+\eta_2+\theta_{14}} + \mathrm{e}^{\xi_2+\eta_1+\theta_{23}} + \mathrm{e}^{\xi_2+\eta_2+\theta_{24}} + \mathrm{e}^{\xi_1+\xi_2+\eta_1+\eta_2+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{23}+\theta_{24}+\theta_{34}}} \tag{36}$$

$$r = \frac{\mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\xi_1+\eta_1+\eta_2+\theta_{13}+\theta_{14}+\theta_{34}} + \mathrm{e}^{\xi_2+\eta_1+\eta_2+\theta_{23}+\theta_{24}+\theta_{34}}}{1+\mathrm{e}^{\xi_1+\eta_1+\theta_{13}} + \mathrm{e}^{\xi_1+\eta_2+\theta_{14}} + \mathrm{e}^{\xi_2+\eta_1+\theta_{23}} + \mathrm{e}^{\xi_2+\eta_2+\theta_{24}} + \mathrm{e}^{\xi_1+\xi_2+\eta_1+\eta_2+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{23}+\theta_{24}+\theta_{34}}} \tag{37}$$

Similarly, we can obtain three-soliton solutions of eq. (1). In this case, g, h, and f in eq. (6) are determined:

$$\begin{aligned}
g = {} & \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} + \mathrm{e}^{\xi_3} + \mathrm{e}^{\xi_1+\xi_2+\xi_3+\eta_1+\eta_2+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{15}+\theta_{23}+\theta_{24}+\theta_{25}+\theta_{34}+\theta_{35}+\theta_{45}} + \\
& + \mathrm{e}^{\xi_1+\xi_2+\xi_3+\eta_1+\eta_3+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{16}+\theta_{23}+\theta_{24}+\theta_{26}+\theta_{34}+\theta_{36}+\theta_{46}} + \\
& + \mathrm{e}^{\xi_1+\xi_2+\xi_3+\eta_2+\eta_3+\theta_{12}+\theta_{13}+\theta_{15}+\theta_{16}+\theta_{23}+\theta_{25}+\theta_{26}+\theta_{35}+\theta_{36}+\theta_{56}} + \\
& + \mathrm{e}^{\xi_1+\xi_2+\eta_1+\theta_{12}+\theta_{14}+\theta_{24}} + \mathrm{e}^{\xi_1+\xi_2+\eta_2+\theta_{12}+\theta_{15}+\theta_{25}} + \mathrm{e}^{\xi_1+\xi_2+\eta_3+\theta_{12}+\theta_{16}+\theta_{26}} + \\
& + \mathrm{e}^{\xi_1+\xi_3+\eta_1+\theta_{13}+\theta_{14}+\theta_{34}} + \mathrm{e}^{\xi_1+\xi_3+\eta_2+\theta_{13}+\theta_{15}+\theta_{35}} + \mathrm{e}^{\xi_1+\xi_3+\eta_3+\theta_{13}+\theta_{16}+\theta_{36}} + \\
& + \mathrm{e}^{\xi_2+\xi_3+\eta_1+\theta_{23}+\theta_{24}+\theta_{34}} + \mathrm{e}^{\xi_2+\xi_3+\eta_2+\theta_{23}+\theta_{25}+\theta_{35}} + \mathrm{e}^{\xi_2+\xi_3+\eta_3+\theta_{23}+\theta_{26}+\theta_{36}}
\end{aligned} \tag{38}$$

$$\begin{aligned}
h = {} & \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_3} + \mathrm{e}^{\xi_1+\xi_2+\eta_1+\eta_2+\eta_3+\theta_{12}+\theta_{14}+\theta_{15}+\theta_{16}+\theta_{24}+\theta_{25}+\theta_{26}+\theta_{45}+\theta_{46}+\theta_{56}} + \\
& + \mathrm{e}^{\xi_1+\xi_3+\eta_1+\eta_2+\eta_3+\theta_{13}+\theta_{14}+\theta_{15}+\theta_{16}+\theta_{34}+\theta_{35}+\theta_{36}+\theta_{45}+\theta_{46}+\theta_{56}} + \\
& + \mathrm{e}^{\xi_2+\xi_3+\eta_1+\eta_2+\eta_3+\theta_{23}+\theta_{24}+\theta_{25}+\theta_{26}+\theta_{34}+\theta_{35}+\theta_{36}+\theta_{45}+\theta_{46}+\theta_{56}} + \\
& + \mathrm{e}^{\xi_1+\eta_1+\eta_2+\theta_{14}+\theta_{15}+\theta_{45}} + \mathrm{e}^{\xi_2+\eta_1+\eta_2+\theta_{24}+\theta_{25}+\theta_{45}} + \mathrm{e}^{\xi_2+\eta_1+\eta_3+\theta_{24}+\theta_{26}+\theta_{46}} + \mathrm{e}^{\xi_3+\eta_1+\eta_3+\theta_{34}+\theta_{36}+\theta_{46}} + \\
& + \mathrm{e}^{\xi_1+\eta_2+\eta_3+\theta_{15}+\theta_{16}+\theta_{56}} + \mathrm{e}^{\xi_2+\eta_2+\eta_3+\theta_{25}+\theta_{26}+\theta_{56}} + \mathrm{e}^{\xi_3+\eta_2+\eta_3+\theta_{35}+\theta_{36}+\theta_{56}}
\end{aligned} \tag{39}$$

$$\begin{aligned}
f = {} & 1 + \mathrm{e}^{\xi_1+\eta_1+\theta_{14}} + \mathrm{e}^{\xi_1+\eta_2+\theta_{15}} + \mathrm{e}^{\xi_1+\eta_3+\theta_{16}} + \mathrm{e}^{\xi_2+\eta_1+\theta_{24}} + \mathrm{e}^{\xi_2+\eta_2+\theta_{25}} + \mathrm{e}^{\xi_2+\eta_3+\theta_{26}} + \mathrm{e}^{\xi_3+\eta_1+\theta_{34}} + \mathrm{e}^{\xi_3+\eta_2+\theta_{35}} + \\
& + \mathrm{e}^{\xi_3+\eta_3+\theta_{36}} + \mathrm{e}^{\xi_1+\xi_2+\eta_1+\eta_2+\theta_{12}+\theta_{14}+\theta_{15}+\theta_{24}+\theta_{25}+\theta_{45}} + \mathrm{e}^{\xi_1+\xi_3+\eta_1+\eta_3+\theta_{13}+\theta_{14}+\theta_{16}+\theta_{34}+\theta_{36}+\theta_{46}} + \\
& + \mathrm{e}^{\xi_1+\xi_3+\eta_2+\eta_3+\theta_{13}+\theta_{15}+\theta_{16}+\theta_{35}+\theta_{36}+\theta_{56}} + \mathrm{e}^{\xi_2+\xi_3+\eta_1+\eta_2+\theta_{23}+\theta_{24}+\theta_{25}+\theta_{34}+\theta_{35}+\theta_{45}} + \\
& + \mathrm{e}^{\xi_2+\xi_3+\eta_1+\eta_3+\theta_{23}+\theta_{24}+\theta_{26}+\theta_{34}+\theta_{36}+\theta_{46}} + \mathrm{e}^{\xi_2+\xi_3+\eta_2+\eta_3+\theta_{23}+\theta_{25}+\theta_{26}+\theta_{35}+\theta_{36}+\theta_{56}} + \\
& + \mathrm{e}^{\xi_1+\xi_2+\xi_3+\eta_1+\eta_2+\eta_3+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{15}+\theta_{16}+\theta_{23}+\theta_{24}+\theta_{25}+\theta_{26}+\theta_{34}+\theta_{35}+\theta_{36}+\theta_{45}+\theta_{46}+\theta_{56}}
\end{aligned} \tag{40}$$

where

$$\xi_i = k_i x - \sum_{j=0}^{3} (-1)^j k_i^j \int a_j(t)\mathrm{d}t + \xi_i^0, \quad \eta_i = l_i x + \sum_{j=0}^{3} l_i^j \int a_j(t)\mathrm{d}t + \eta_i^0, \quad (i = 1,\ 2,\ 3) \tag{41}$$

$$\mathrm{e}^{\theta_{ij}} = -(k_i - k_j)^2, \quad \mathrm{e}^{\theta_{(i+n)(j+n)}} = -(l_i - l_j)^2, \quad (i < j = 2,\ 3) \tag{42}$$

$$e^{\theta_{i(j+n)}} = -\frac{1}{(k_i + l_j)^2}, \quad (i, j = 1, 2, 3) \tag{43}$$

By induction, we can finally reach the n-soliton solutions determined by eqs. (6) and eqs. (10)-(14) of eq. (1). Thus, we finish the proof of *Theorem 2*.

Conclusion

In summary, we have extended Hirota's bi-linear method [2] to a new gAKNS hierarchy in eq. (1). One of the key steps in the procedure of extending Hirota's bi-linear method [2] is to reduce eq. (1) to the bi-linear form in eqs. (7) and (8) by the transformation eq. (6) introduced in this paper. As a result, exact and explicit one-soliton solutions, two-soliton solutions, three-soliton solutions and the uniform formulae of n-soliton solutions of the new gAKNS hierarchy are obtained. The continuous functions $\alpha_1(t)$, $\alpha_2(t)$,..., and $\alpha_n(t)$ in the obtained solutions provide enough freedom for us to describe enrich structures of these soliton solutions.

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