AN ANALYTICAL SOLUTION OF FRACTIONAL BURGERS EQUATION

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Using the fractional complex transform, the fractional partial differential equations can be reduced to ordinary differential equations which can be solved by the auxiliary equation method. Non-linear superposition formulation of Riccati equation is applied, and a complex infinite sequence solution is obtained.

Key words: *fractional partial differential equation, complex infinite sequence solution, non-linear superposition formula*

Introduction

In recent years, with the development of fractional calculus theory, the application of fractional calculus has become more and more extensive in nature and life. In the study, it was found in many areas, because of the fractional derivative has good hereditary, memory and non-locality, some physical processes and dynamics of the system in nature and life have been described more effective and accurate [1-3]. Such as the earthquake model and the fluid dynamic traffic model can achieve a better effect after the introduction of fractional derivative.

As an extension of the traditional model, the fractional partial differential equations based on the fractional calculus theory also get more and more applications in different fields such as fractal dynamics, chaos, and so on. This kind of equations is more and more widely used in the modeling of different fields, such as the boundary layer effect, the viscoelastic material [4-6], the anomalous diffusion of ions in the nerve cell [7], and so on. In this case, how to solve the exact solutions of fractional partial differential equation becomes a hotspot. Recently, some methods has been proposed, for example, fractional Riccati auxiliary equation method, fractional G′/G method, variable iteration method, *etc.*

Mathematical approach

Definition and properties of fractional derivative

First of all, we use the modified Riemann-Liouville derivative given by Jumarie [8]:

$$\mathrm{D}_x^\alpha f(x)=\begin{cases}\dfrac{1}{\Gamma(-\alpha)}\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_0^x(x-\xi)^{-\alpha-1}[f(\xi)-f(0)]\mathrm{d}\xi, & \alpha<0\\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_0^x(x-\xi)^{-\alpha}[f(\xi)-f(0)]\mathrm{d}\xi, & 0<\alpha<1\\ [f^{(n)}(x)]^{\alpha-n}, \quad n\le\alpha\le n+1, \quad n\ge 1\end{cases} \tag{1}$$

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Some important properties are:

$$\mathrm{D}_t^\alpha t^\alpha = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}, \qquad \gamma > 0 \tag{1a}$$

$$\mathrm{D}_t^\alpha [f(t)g(t)] = g(t)\mathrm{D}_t^\alpha f(t) + f(t)\mathrm{D}_t^\alpha g(t) \tag{1b}$$

$$\mathrm{D}_t^\alpha f[g(t)] = f_g'[g(t)]\mathrm{D}_t^\alpha g(t) = \mathrm{D}_g^\alpha f[g(t)][g'(t)]^\alpha \tag{1c}$$

Description of the Riccati expansion method

The second, we use the Riccati equation:

$$\phi' = R + \phi^2 \tag{2}$$

It has three kinds of solutions [9]:

$$\phi(\xi) = \begin{cases} -\sqrt{-R}\tanh(\sqrt{-R}\xi), & \text{or} \quad -\sqrt{-R}\coth(\sqrt{-R}\xi), \qquad R<0 \\ \sqrt{R}\tanh(\sqrt{R}\xi), & \text{or} \quad -\sqrt{R}\coth(\sqrt{R}\xi), \qquad R>0 \\ -\dfrac{1}{\xi}, \quad R=0 \end{cases} \tag{2a}$$

Taogetusang *et al.* [10] has been calculated and obtained the following solution:

$$\phi(\xi) = \frac{BR + A\sqrt{-R}\tanh(\sqrt{-R}\xi)}{-A + B\sqrt{-R}\tanh(\sqrt{-R}\xi)}, \qquad R<0 \tag{2b}$$

$$\phi(\xi) = \frac{-(r\sqrt{R}+CR)\cos(\sqrt{R}\xi) + (r - C\sqrt{R})\sqrt{R}\sin(\sqrt{R}\xi)}{(r - C\sqrt{R})\cos(\sqrt{R}\xi) + (r + C\sqrt{R})\sin(\sqrt{R}\xi)}, \qquad R>0 \tag{2c}$$

$$\phi(\xi) = \frac{-3BR + 4A\sqrt{R} - 5BR\sin(2\sqrt{R}\xi) - 5A\sqrt{R}\cos(2\sqrt{R}\xi)}{3A + 4B\sqrt{R} + 5A\sin(2\sqrt{R}\xi) - 5B\sqrt{R}\cos(2\sqrt{R}\xi)}, \qquad R>0 \tag{2d}$$

$$\phi(\xi) = \frac{-BR + A\sqrt{R}[\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]}{A + B\sqrt{R}[\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]}, \qquad R>0 \tag{2e}$$

$$\phi(\xi) = \frac{\sqrt{R}[\cos(\sqrt{R}\xi) + \sin(\sqrt{R}\xi)]}{\cos(\sqrt{R}\xi) - \sin(\sqrt{R}\xi)}, \qquad R>0 \tag{2f}$$

$$\phi(\xi) = \frac{\sqrt{R}\{-2AB\sqrt{R} + (A^2 - B^2R)[\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]\}}{A^2 - B^2R + 2AB\sqrt{R} - [\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]}, \qquad R>0 \tag{2g}$$

where r, A, B, and C are arbitrary constants that are not all zero.

Description of the method

Consider the fractional partial differential equation in the form:

$$F(u,u_t,u_x,u_y,u_z,\cdots,\mathrm{D}_t^{\alpha}u,\mathrm{D}_x^{\beta}u,\mathrm{D}_y^{\gamma}u,\mathrm{D}_z^{\delta}u,\cdots)=0,\quad 0<\alpha,\beta,\gamma,\delta\le 1 \tag{3}$$

Step 1. The fractional complex transform reads:

$$u(x,y,z,\cdots,t)=u(\xi) \tag{3a}$$

$$\xi=\frac{Bx^{\beta}}{\Gamma(1+\beta)}+\frac{Cx^{\gamma}}{\Gamma(1+\gamma)}+\frac{Dx^{\delta}}{\Gamma(1+\delta)}+\cdots+\frac{Ax^{\alpha}}{\Gamma(1+\alpha)} \tag{3b}$$

where A, B, C, and D are arbitrary constants. By using the traveling wave transformation and the properties of fractional derivative, the equation convert into an ordinary differential equation (ODE) of the form:

$$P(u,u',u'',u''',\cdots)=0 \tag{3c}$$

Step 2. Suppose that the solution of the eq. (3), can be expressed:

$$u(\xi)=\sum_{i=0}^{m}a_i\phi^i(\xi) \tag{3d}$$

where

$$\phi(\xi)=R+\phi^2(\xi) \tag{3e}$$

Step 3. Balance the highest order derivative term and non-linear term in the equation, m can be calculated. Substitution into the equations of a_i are obtained.

Step 4. Using the MATHEMATICA, we can obtain a_i, according to a_i, (3b) solutions of (3e), the solution of eq. (3) can be obtained.

Application

We apply the method to the time fractional Burgers equation [11, 12] in the form:

$$\mathrm{D}_t^{\alpha}u+\varepsilon uu_x-\nu u_{xx}=0,\qquad t>0,\qquad 0<\alpha\le 1 \tag{4}$$

where α is a parameter describing the order of the fractional time derivative.

Step 1. The only time fractional derivative equations, so the *fractional complex transform*:

$$u=u(\xi)\quad \xi=Mx+N\frac{t^{\alpha}}{\Gamma(1+\alpha)} \tag{5}$$

where M and N are constants.

By substituting eq. (5) into eq. (4), eq. (4) is reduced into an ODE in the form:

$$Nu'+\varepsilon Muu'-\nu M^2u''=0 \tag{6}$$

By integrating once, eq. (6) convert into:

$$Nu+\frac{1}{2}\varepsilon Mu^2-\nu M^2u'+c=0 \tag{7}$$

Step 2. Suppose that the solution of the equation can be expressed:

$$u(\xi)=\sum_{i=0}^{m}a_i\phi^i(\xi) \tag{8}$$

where $\varphi(\xi)$ is the solution of Riccati eq. (2).

Step 3. Balance uu' and u''. It can be known that $m = 1$. So the solution of the eq. (4) is:

$$u(\xi)=a_0+a_1\phi(\xi) \tag{9}$$

Substituting eq. (9) into eq. (6), collecting coefficient of φ and φ^2 to zero, Get the following equations:

$$\begin{gathered}Na_1+\varepsilon Ma_0a_1=0\\ \frac{1}{2}\varepsilon Ma_1^2-\nu M^2a_1=0\\ Na_0+\frac{1}{2}\varepsilon Ma_0^2-\nu M^2Ra_1+c=0\end{gathered} \tag{10}$$

Step 4. Using the MATHEMATICA, we can obtain a_i:

$$a_0=-\frac{N}{\varepsilon M},\quad a_1=-\frac{2\nu M}{\varepsilon},\quad 2c\varepsilon M=N^2+4\nu^2M^4R,\qquad \varepsilon\neq 0,\quad M\neq 0 \tag{11}$$

So, the solution of eq. (4) is:

$$u(\xi)=-\frac{N}{\varepsilon M}+-\frac{2\nu M}{\varepsilon}\phi(\xi) \tag{12}$$

Next, we use the non-linear superposition formulas of Riccati equation to create the infinite sequence solutions of the equation and the new exact solutions of the equation.

– substituting the following non-linear superposition formula and eq. (5) into eq. (12), infinite sequence solutions of hyperbolic function can be created:

$$\begin{gathered}\phi_k(\xi)=\frac{-BR+A\varphi_{k-1}(\xi)}{A+B\varphi_{k-1}(\xi)},\qquad k=1,2,\cdots\\ \phi_0(\xi)=\frac{BR+A\sqrt{-R}\tanh(\sqrt{-R}\xi)}{-A+B\sqrt{-R}\tanh(\sqrt{-R}\xi)},\qquad R<0\end{gathered} \tag{13}$$

When $k = 1$, a new exact solution of hyperbolic function can be obtained:

$$\begin{gathered}u(\xi)=-\frac{N}{\varepsilon M}-\frac{2\nu M[2BR-(A^2-B^2)R\sqrt{-R}\tanh(\sqrt{-R}\xi)]}{\varepsilon\{2BR-[-A^2+B^2R+2AB\sqrt{-R}\tanh(\sqrt{-R}\xi)]\}}\\ R<0,2c\varepsilon M=N^2+4\nu^2M^4R,\qquad \varepsilon\neq 0,\qquad M\neq 0\end{gathered} \tag{14}$$

– substituting the following non-linear superposition formula and eq. (5) into eq. (12), infinite sequence solutions of triangular function can be created:

$$\phi_k(\xi)=\frac{-BR+A\varphi_{k-1}(\xi)}{A+B\varphi_{k-1}(\xi)},\qquad k=1,2,\cdots$$
$$\phi_0(\xi)=\frac{\sqrt{R}\{-2AB\sqrt{R}+(A^2-B^2R)[\sec(2\sqrt{R}\xi)+\tan(2\sqrt{R}\xi)]\}}{A^2-B^2R+2AB\sqrt{R}[\sec(2\sqrt{R}\xi)+\tan(2\sqrt{R}\xi)]},\qquad R>0 \tag{15}$$

When $k=1$, a new exact solution of triangular function can be obtained:

$$u(\xi)=-\frac{N}{\varepsilon M}-\frac{2\nu M}{\varepsilon}\frac{B^3R^2-3A^2BR+A\sqrt{R}(2B+A^2-B^2R)[\sec(2\sqrt{R}\xi)+\tan(2\sqrt{R}\xi)]}{A^3-3AB^2R+B\sqrt{R}(3A^2-B^2R)[\sec(2\sqrt{R}\xi)+\tan(2\sqrt{R}\xi)]} \tag{16}$$
$$R<0,\qquad 2c\varepsilon M=N^2+4\nu^2M^4R,\qquad \varepsilon\neq 0,\qquad M\neq 0$$

– substituting the following non-linear superposition formula and eq. (5) into eq. (12), complex infinite sequence solutions of hyperbolic functions with rational function can be created:

$$\phi_k(\xi)=\frac{iR[im\sqrt{R}+(m+iD\sqrt{R}+CR)\varphi_{k-1}(\xi)+[-CR+D\varphi_{k-1}(\xi)]\varphi_{k-2}(\xi)]}{-\sqrt{R^3}[D+C\varphi_{k-1}(\xi)]+[m\sqrt{R}+iDR+C\sqrt{R^3}-im\varphi_{k-1}(\xi)]\varphi_{k-2}(\xi)}$$
$$\phi_0(\xi)=-\sqrt{-R}\tanh(\sqrt{-R}\xi),\qquad \varphi_1(\xi)=-\frac{1}{\xi},\qquad k=2,3,\cdots \tag{17}$$

When $k=2$, a new complex exact solution of hyperbolic functions with rational function can be obtained:

$$u(\xi)=-\frac{N}{\varepsilon M}-\frac{2\nu M}{\varepsilon}\frac{m^2\left\{\sqrt{\frac{m^2}{D^2}}(D-m\xi)+m\left[-1+\tanh\left(\frac{m}{D}\xi\right)\right]\right\}}{D^3\sqrt{\left(\frac{m^2}{D^2}\right)^3}\xi+\left[Dm^2-m^3\xi+D^3\left(\frac{m}{D}\right)^3\right]\tanh\left(\frac{m}{D}\xi\right)} \tag{18}$$
$$mD<0,\qquad 2c\varepsilon M=N^2+4\nu^2M^4R,\qquad \varepsilon\neq 0,\qquad M\neq 0$$

– substituting the following non-linear superposition formula and eq. (5) into eq. (12), infinite sequence solutions of hyperbolic functions with rational function can be created:

$$\phi_k(\xi)=\frac{R[-r\phi_{k-3}(\xi)+(p+r)\phi_{k-2}(\xi)-p\phi_{k-1}(\xi)]}{-r\phi_{k-2}(\xi)\phi_{k-1}(\xi)+\phi_{k-2}(\xi)[-p\phi_{k-2}(\xi)+(p+r)\phi_{k-1}(\xi)]},\qquad k=3,4,\cdots$$
$$\phi_0(\xi)=-\sqrt{-R}\tanh(\sqrt{-R}\xi),\qquad \phi_1(\xi)=\frac{BR+A\sqrt{-R}\tanh(\sqrt{-R}\xi)}{-A+B\sqrt{-R}\tanh(\sqrt{-R}\xi)},\qquad \phi_2(\xi)=-\frac{1}{\xi} \tag{19}$$

When $k=3$, a new complex exact solution of hyperbolic functions with rational function can be obtained:

$$u(\xi)=-\frac{N}{\varepsilon M}+\frac{2\nu MR[-A+\sqrt{-R}(B+A\xi)\tanh(\sqrt{-R}\xi)+BR\xi\tanh^2(\sqrt{-R}\xi)]}{\varepsilon[BR-\sqrt{-R}(-A+BR\xi)\tanh(\sqrt{-R}\xi)+AR\xi\tanh^2(\sqrt{-R}\xi)]} \tag{20}$$
$$2c\varepsilon M=N^2+4\nu^2M^4R,\qquad \varepsilon\neq 0,\qquad M\neq 0$$

– substituting the following non-linear superposition formula and eq. (5) into eq. (12), infinite sequence solutions of triangular functions with rational function can be created:

$$\phi_k(\xi) = \frac{R[-r\phi_{k-3}(\xi) + (p+r)\phi_{k-2}(\xi) - p\phi_{k-1}(\xi)]}{-r\phi_{k-2}(\xi)\phi_{k-1}(\xi) + \phi_{k-2}(\xi)[-p\phi_{k-2}(\xi) + (p+r)\phi_{k-1}(\xi)]}, \qquad k = 3, 4, \cdots$$

$$\phi_0(\xi) = \frac{\sqrt{R}[\cos(\sqrt{R}\xi) + \sin(\sqrt{R}\xi)]}{\cos(\sqrt{R}\xi) - \sin(\sqrt{R}\xi)} \tag{21}$$

$$\phi_1(\xi) = \frac{\sqrt{R}\{-2AB\sqrt{R} + (A^2 - B^2R)[\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]\}}{A^2 - B^2R + 2AB\sqrt{R}[\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]}, \qquad \phi_2(\xi) = -\frac{1}{\xi}$$

When $k = 3$, a new complex exact solution of triangular functions with rational function can be obtained:

$$u(\xi) = -\frac{N}{\varepsilon M} + \frac{2\nu M}{\varepsilon}\frac{\sqrt{R}[\eta_1(\xi) + \eta_2(\xi)\cos(2\sqrt{R}\xi) + \eta_3(\xi)\sin(2\sqrt{R}\xi)]}{\eta_4(\xi) - \eta_3(\xi)\cos(2\sqrt{R}\xi) + \eta_2(\xi)\sin(2\sqrt{R}\xi)}$$

$$\eta_1(\xi) = -A^2 + B^2R - 2ABR\xi, \qquad \eta_2(\xi) = \sqrt{R}(-2AB - A^2\xi + B^2R\xi) \tag{22}$$

$$\eta_3(\xi) = A^2 - B^2R - 2ABR\xi, \qquad \eta_4(\xi) = \sqrt{R}(2AB - A^2\xi + B^2R\xi)$$

$$2c\varepsilon M = N^2 + 4\nu^2M^4R, \qquad \varepsilon \neq 0, \qquad M \neq 0$$

Conclusion

In this paper, non-linear superposition formulas of Riccati equation have been used in fractional partial differential equations first time. We successfully obtained the complex exact solutions and complex infinite sequence solutions of the time fractional Burgers equation. After introducing the complex traveling wave transformation, the fractional partial differential equation can be reduce to an ODE. Visible, this is an effective means to all the fractional partial differential equations. In future research, we could suppose the solutions can be expressed by other forms, which we can create more interesting solutions.

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