A SHORT REVIEW ON ANALYTICAL METHODS FOR FRACTIONAL EQUATIONS WITH HE'S FRACTIONAL DERIVATIVE

by

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He's fractional derivative is adopted in this paper, and analytical methods for fractional differential equations are briefly reviewed, two modifications of the exp-function method (the generalized Kudryashov method and generalized exponential rational function method) are emphasized, and fractional Benjamin-Bona-Mahony equation with He's fractional derivative is used as an example to elucidate the solution process.

Key words: *He's fractional derivative, generalized Kudryashov method, generalized exponential rational function method, fractional Benjamin-Bona-Mahony equation*

Introduction

The investigation of non-linear partial differential equations (PDE) has made enormous advances in a large number of physical, engineering, mathematical, and other problems. In recent years, fractional differential equations (FDE) [1] have attracted much interest, since many processes in biology, physics, engineering, mathematics, economics, and other areas of science can be exactly described by fractional calculus, especially thermal science for discontinuous media has to adopt fractional models [2-4]. Consequently, the study in finding solutions for FDE plays an important role in scientific research. At present, many powerful methods for finding solutions of FDE have been put forward, for examples, the G′/G-expansion method [5], the modified simple equation method [5], the extended Jacobi's elliptic functions expansion method [6], among which the exp-function method [7] seems to be more suitable for fractional calculus, and its modifications, the generalized Kudryashov (GK) method [8] and generalized exponential rational function (GERF) method [9], have also been caught much attention recently.

Following are the common definitions for fractional derivatives.

Riemann-Liouville derivative is defined [1, 3]:

$$\mathrm{D}_x^{\alpha}[u(x)] = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^x (x-t)^{n-\alpha-1} f(t)\mathrm{d}t \tag{1}$$

Caputo derivative is defined [1, 3]:

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$$\mathrm{D}_x^{\alpha}[u(x)] = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{\mathrm{d}^n u(t)}{\mathrm{d}t^n} \mathrm{d}t \tag{2}$$

Yang local fractional derivative is defined [1-3, 10-14]:

$$\mathrm{D}_x^{\alpha}[u(x_0)] = \lim_{x \to x_0} \frac{\Delta^{\alpha}[u(x) - u(x_0)]}{x - x_0} \tag{3}$$

where $\Delta^{\alpha}[u(x) - u(x_0)] \cong \Gamma(1+\alpha)\Delta[u(x) - u(x_0)]$.

Yang-Srivastava-Machado fractional derivative [15, 16]:

$${}^{\mathrm{YSM}}\mathrm{D}_{a^+}^{(\alpha)} u(x) = \frac{\Re(\alpha)}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \exp\left[-\frac{\alpha}{1-\alpha}(x-\lambda)\right] u(\lambda)\mathrm{d}\lambda \tag{4}$$

where $a \le x$, $\alpha(0 < \alpha < 1)$ is a real number, and $\Re(\alpha)$ is a normalization function depending on α such that $\Re(0) = \Re(1) = 1$.

More recently, He's fractional derivative [3, 17, 18], derived from the variational iteration algorithm, is given:

$$\mathrm{D}_t^{\alpha} u = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^t (s-t)^{n-\alpha-1}[u_0(s) - u(s)]\mathrm{d}s \tag{5}$$

where $u_0(s)$ is a known function.

In fact, He's fractional derivative is a completely new approach to heat-conduction and porous media problems [3, 17-21]. The main target of this paper is to extend the exp-function method and to suggest the two modifications, *e. g.*, the GK and GERF methods to obtain the exact solutions of the space-time fractional Benjamin-Bona-Mahony (BBM) equation within He's fractional derivative.

Fractional BBM equation

This paper suggests an improvement of the BBM equation which reads:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^2 \partial t} = 0 \tag{6}$$

The BBM equation is also known as the regularized long-wave equation, which is an improvement of KdV equation for modeling long surface gravity waves of small amplitude. This equation is valid only for continuum medium. When the flow surface is covered with porous medium, which is designed to control the wave morphology, eq. (6) has to be modified using He's fractional derivative:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + \frac{\partial^{\beta} u}{\partial x^{\beta}} + u\frac{\partial^{\beta} u}{\partial x^{\beta}} - \frac{\partial^{2\beta}\partial^{\alpha} u}{\partial x^{2\beta} \partial t^{\alpha}} = 0, \qquad 0 < \alpha, \qquad \beta \le 1 \tag{7}$$

This fractional partner of BBM equation behaves in a way that BBM equation does not, and has practical applications to control surface gravity waves, we will discuss some special properties of eq. (7) in the forthcoming section.

Equation (7) is extremely difficult to be solved using some known methods in open literature, this paper shows that the exp-function method and its modifications are extremely effective for eq. (7).

Exp-function method

Let us consider the following PDE in the form [7, 22]:

$$F(u, u_t, u_x, u_{xx}, u_{xxx}, ...) = 0$$

Making use of the travelling wave transform $\eta = px + qt$, where p and q are constants, we have:

$$\tilde{F}(u, u', u'', u''', ...) = 0$$

According to the exp-function method [7], the solution can be expressed in the form:

$$u(\eta) = \frac{\sum_{n=-r}^{s} a_n \exp(n\eta)}{\sum_{m=-\omega}^{v} b_m \exp(m\eta)} \tag{8}$$

where a_n and b_m are the unknown constants and r, s, ω, and v are the positive integers which are further determined. To determine the values of r and ω, we balance the linear term of highest order in eq. (7) with the highest order non-linear term. Similarly, to determine the values of s and v, we balance the linear term of lowest order in eq. (7) with the lowest order non--linear term.

We have noticed the method is adopted to extend its applications to local fractional calculus. In next section, the idea to apply to find the solutions for the FDE involving He's fractional derivative is suggested.

Two modifications

We consider the following general fractional differential equation:

$$f(u,\ u_t^{\alpha},\ u_x^{\beta},\ u_y^{\gamma},\ u_z^{\lambda},\ u_t^{2\alpha},\ u_x^{2\beta},\ u_y^{2\gamma},\ u_z^{2\lambda},\ ...) = 0 \tag{9}$$

where $0 < \alpha, \beta, \gamma, \lambda \leq 1$, and u_t^{α} is He's fractional derivative.

Li and He [23], He and Li [24], and Li *et al.* [25] presented the fractional complex transformation to convert FDE into ordinary differential equations (ODE) or PDE. Following the idea, we can structure the following fractional complex transformation:

$$u(x, t) = U(\zeta) \tag{10}$$

$$\zeta = \frac{\kappa x^{\beta}}{\Gamma(\beta + 1)} - \frac{\tau t^{\alpha}}{\Gamma(\alpha + 1)} \tag{11}$$

where κ and τ are non-zero constants. We can reduce eq. (9) into the ODE:

$$\tilde{f}(U,\ U',\ U'',\ U''',\ ...) = 0, \tag{12}$$

where $U' = \mathrm{d}U/\mathrm{d}\zeta$, $U'' = \mathrm{d}^2U/\mathrm{d}\zeta^2$, … Now we consider two modifications of the exp-function method for the FDE.

The GK method involving He's fractional derivative

We assume that the exact solution of eq. (12) can be expressed in the form [8]:

$$U(\zeta)=\frac{\sum_{i=0}^{N}a_i\varphi^i(\zeta)}{\sum_{j=0}^{M}b_j\varphi^j(\zeta)} \tag{13}$$

where a_i (i = 0, 1, 2, …, N), b_i (j = 0, 1, 2, …, M) are constants to be determined later, and:

$$\varphi(\zeta)=\frac{1}{1\pm a^{\zeta}} \tag{14}$$

which satisfies:

$$\varphi_{\zeta}(\zeta)=[\varphi^2(\zeta)-\varphi(\zeta)]\ \ln\ a \tag{15}$$

Balancing the highest order linear term with the highest order non-linear term in eq. (12), we can receive some values of N and M.

Substituting eq. (13) into eq. (12) yields an equation involving the function $\varphi(\zeta)$ and its derivatives. Then putting eq. (15) and the various derivatives of $\varphi_{\zeta}(\zeta)$ into the equation just obtained provides a polynomial $R[\varphi(\zeta)]$. Finally, collecting all terms with the same powers of $\varphi(\zeta)$ and equating each coefficient of $R[\varphi(\zeta)]$ to zero, we can obtain a system of algebraic equations. Solving this algebraic equations system and subsequently substituting these values into eq. (13), we can attain the exact solutions of eq. (9) immediately.

The GERF method involving He's fractional derivative

We suppose that the exact solution of eq. (12) can be expressed as [9]:

$$U(\zeta)=\sum_{i=0}^{L}d_i\varphi(\zeta)^i=\sum_{i=0}^{L}\frac{d_i}{(1\pm a^{\zeta})^i} \tag{16}$$

where d_i = (i = 0, 1, …. N) are constants to be determined later. Substituting the derivatives of function $U(\xi)$ along with eq. (16) into eq. (12) gives the polynomial equation:

$$\Psi[\varphi(\zeta)]=0 \tag{17}$$

Then, collecting all terms with the same powers of $\varphi(\zeta)$ and equating each coefficient of $\Psi[\varphi(\zeta)]$ to zero, it follows a set of algebraic equations. Finally, solving this algebraic equations system and subsequently substituting these value into eq. (16), we can obtain the exact solutions of eq. (9) instantly.

In this section, two modifications of the exp-function method [7] for the FDE, *e. g.*, the GK and GERF methods, are presented based on He's fractional derivative.

Following eqs. (10) and (11), we similarly have the fractional complex transformation [3]:

$$\begin{cases} u(x,t)=U(\xi), \\ \xi=\dfrac{x^{\beta}}{\Gamma(\beta+1)}-\dfrac{\tau t^{\alpha}}{\Gamma(\alpha+1)} \end{cases} \tag{18}$$

where τ is a constant, such that eq. (17) takes the following form:

$$(1-\tau)U'+UU'+\tau U''=0 \tag{19}$$

Solutions for the space-time fractional BBM equation via the GK method involving He's fractional derivative

By balancing the highest order derivative term U''' and non-linear term UU' in eq. (19), we receive $N - M + 3 = 2N - 2M + 1$.

If we choose $M = 1$, then $N = 3$. Thus, eq. (13) becomes:

$$U(\xi)=\frac{a_0+a_1\varphi(\xi)+a_2\varphi(\xi)^2+a_3\varphi(\xi)^3}{b_0+b_1\varphi(\xi)} \tag{20}$$

Substituting eq. (20) along with eq. (15) into eq. (19) and then setting the coefficients of $\varphi(\xi)^i (i = 1, 2, \ldots, 9)$ to zero, we get a series of algebraic equations with respect to a_0, a_1, a_2, a_3, b_0, and b_1. Solving the algebraic equations by MAPLE, we have the following:

$$\begin{gathered} a_0=-b_0+\tau b_0-\tau b_0(\ln a)^2, \quad a_1=-\tau(\ln a)^2 b_1+12\tau b_0(\ln a)^2+\tau b_1-b_1, \\ a_2=12\tau(\ln a)^2 b_1-12\tau b_0(\ln a)^2, \quad a_3=-12\tau(\ln a)^2 b_1, \quad b_0=b_0, \quad b_1=b_1 \end{gathered} \tag{21}$$

where $b_0 = b_0$, $b_1 = b_1$ are the arbitrary constants.

If we put eq. (21) along with eq. (14) into eq. (20), we can obtain the following solution of eq. (18):

$$U_1(\xi)=\frac{A_1(\xi)}{B_1(\xi)} \tag{22}$$

and

$$U_2(\xi)=\frac{A_2(\xi)}{B_2(\xi)} \tag{23}$$

where

$$\begin{aligned} A_1(\xi)=&-b_0+\tau b_0-\tau b_0(\ln a)^2+\frac{12\tau b_0(\ln a)^2-\tau(\ln a)^2 b_1+\tau b_1-b_1}{1+a^{\xi}}+ \\ &+\frac{-12\tau b_0(\ln a)^2+12\tau(\ln a)^2 b_1}{(1+a^{\xi})^2}-\frac{12\tau(\ln a)^2 b_1}{(1+a^{\xi})^3} \end{aligned} \tag{24}$$

$$B_1(\xi)=b_0+\frac{b_1}{1+a^{\xi}} \tag{25}$$

$$\begin{aligned} A_2(\xi)=&-b_0+\tau b_0-\tau b_0(\ln a)^2+\frac{12\tau b_0(\ln a)^2-\tau(\ln a)^2 b_1+\tau b_1-b_1}{1-a^{\xi}}+ \\ &+\frac{-12\tau b_0(\ln a)^2+12\tau(\ln a)^2 b_1}{(1-a^{\xi})^2}-\frac{12\tau(\ln a)^2 b_1}{(1-a^{\xi})^3} \end{aligned} \tag{26}$$

$$B_2(\xi)=b_0+\frac{b_1}{1-a^{\xi}} \tag{27}$$

Substituting eq. (24) and eq. (25) into eq. (22), we have:

$$u(x,t)=\frac{a^{2\xi}[-\tau(\ln a)^2-1+\tau]+a^{\xi}[-2+10\tau(\ln a)^2+2\tau]-1-\tau(\ln a)^2+\tau}{(1+a^{\xi})^2} \tag{28}$$

where $\xi = x^{\beta}/\Gamma(\beta+1)-\tau t^{\alpha}/\Gamma(\alpha+1)$.

Similarly, from eqs. (26) and eq. (27), eq. (23) can be written:

$$u(x,t)=\frac{a^{2\xi}[-\tau(\ln a)^2-1+\tau]+a^{\xi}[2-10\tau(\ln a)^2-2\tau]-1-\tau(\ln a)^2+\tau}{(-1+a^{\xi})^2} \tag{29}$$

where $\xi = x^{\beta}/\Gamma(\beta+1)-\tau t^{\alpha}/\Gamma(\alpha+1)$.

Solutions for the space-time fractional BBM equation via the GERF method involving He's fractional derivative

By balancing the highest order derivative term U''' and non-linear term UU' in eq. (19), we receive $L + 3 = 2L + 1$, then $L = 2$. Thus, eq. (16) becomes:

$$U(\xi)=d_0+\frac{d_1}{1\pm a^{\xi}}+\frac{d_2}{(1\pm a^{\xi})^2} \tag{30}$$

Substituting eq. (30) into eq. (19) and then setting the coefficients of $(a^{\xi})^i (i = 1, 2, 3, 4)$ to zero, we receive a series of algebraic equations with respect to, d_0, d_1, and d_2. Solving the algebraic equations by MAPLE, we have:

$$d_0=-1-\tau(\ln a)^2+\tau, \qquad d_1=12\tau(\ln a)^2, \qquad d_2=-12\tau(\ln a)^2 \tag{31}$$

Substituting eq. (31) into eq. (30) gives the following exact solutions of eq. (17):

$$u(x,t)=-1-\tau(\ln a)^2+\tau+\frac{12\tau(\ln a)^2}{1+a^{\xi}}-\frac{12\tau(\ln a)^2}{(1+a^{\xi})^2} \tag{32}$$

$$u(x,t)=-1-\tau(\ln a)^2+\tau+\frac{12\tau(\ln a)^2}{1-a^{\xi}}-\frac{12\tau(\ln a)^2}{(1-a^{\xi})^2} \tag{33}$$

where $\xi = x^{\beta}/\Gamma(\beta+1)-\tau t^{\alpha}/\Gamma(\alpha+1)$.

The graph of eq. (32) is illustrated in fig. 1(a) and the graphs of eq. (33) are demonstrated in figs. 1(b), 1(c), and 1(d) for $a = 10$ and $\tau = 1$.

Remark

In fact, eq. (28) is in agreement with eq. (32). Meanwhile, eq. (29) is in line with eq. (33). Furthermore, compared with the GK method, the GERF method is easier to solve FDE within He's fractional derivative. Actually, the GK and the GERF methods are all derived from the exp-function method [7] involving He's fractional derivative.

Discussion and conclusion

We conclude that the wave morphology can be effectively controlled by the fractional orders involved in the fractional BBM equation as illustrated in the given illustrations. When $\alpha = \beta = 1$, we obtain u_0 given in eq. (6), the cases when $\beta \to 0$ and $\beta \to 1$ implies a solid wall and a continuous flow, respectively.

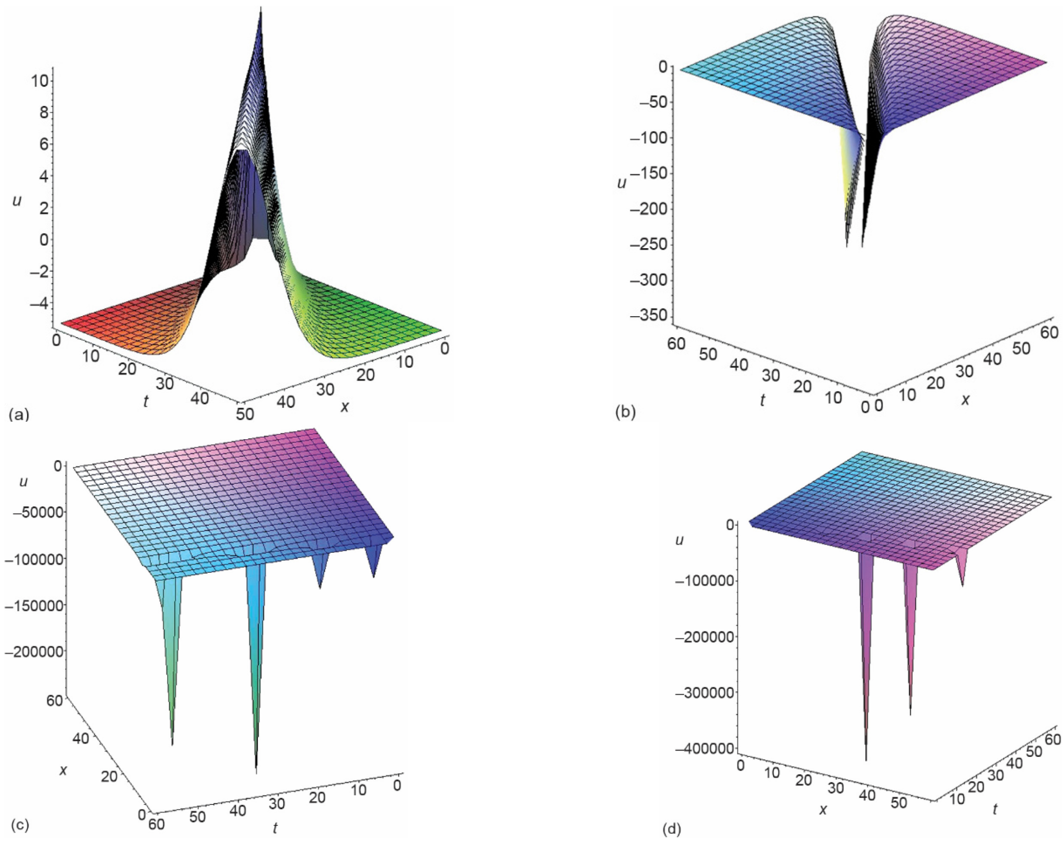

Figure 1. (a) eq. (32) (α = 1/2, β = 1/2), (b) eq. (33) (α = 1/2, β = 1/2), (c) eq. (33) (α = 1/3, β = 1/5), (d) eq. (33) (α = 1/3, β = 1/2)

In the present work, the exp-function method [7] within He's fractional derivative was extended for the first time. The space-time fractional BBM equation was solved by the GK and GERF methods which were two modifications of the exp-function method with He's fractional derivative. The exact solutions were graphically discussed. It is shown that the GERF method is easier than GK method to handle FDE. The performances of the previously mentioned methods are also substantially influential and absolutely reliable for finding new exact solutions of other FDE.

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Nomenclature

x – space co-ordinate, [m]
t – time co-ordinate, [s]
$u(x, t)$ – velocity, [ms^{-1}]
α, β – fractional order, [–]

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