

## HEAT TRANSFER ANALYSIS BASED ON CATTANEO-CHRISTOV HEAT FLUX MODEL AND CONVECTIVE BOUNDARY CONDITIONS FOR FLOW OVER AN OSCILLATORY STRETCHING SURFACE

by

**Sami ULLAH KHAN<sup>a\*</sup>, Nasir ALI<sup>b</sup>, Tasawar HAYAT<sup>c,d</sup>, and Zaheer ABBAS<sup>e</sup>**

<sup>a</sup>Department of Mathematics, COMSATS University Islamabad, Sahiwal, Pakistan

<sup>b</sup>Department of Mathematics and Statistics, International Islamic University, Islamabad, Pakistan

<sup>c</sup>Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

<sup>d</sup>Nonlinear Analysis and Applied Mathematics Research Group,

Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia

<sup>e</sup>Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, Pakistan

Original scientific paper

<https://doi.org/10.2298/TSCI160225172U>

*In this study, we investigate the heat transfer characteristics in unsteady boundary-layer flow of Maxwell fluid by using Cattaneo-Christov heat flux model and convective boundary conditions. The flow is caused by a sheet which is stretched periodically back and forth in its own plane. The physical model that takes into account the effects of constant applied magnetic field is transformed into highly non-linear PDE under boundary-layer approximations. The solution of dimensionless version of these equations is developed using homotopy analysis method. The simulations are presented in the form of temperature and velocity profiles for suitable range of physical parameters. The obtained results illustrate that an increase in Deborah number and Hartmann number suppress the velocity profiles. It is further observed that Cattaneo-Christov heat flux model predicts the suppression of thermal boundary-layer thickness as compared to Fourier law.*

Key words: Maxwell fluid, Cattaneo-Christov heat flux model, oscillatory stretching sheet, homotopy analysis method

### Introduction

The study of convective of heat transfer gained great attention of investigators because of its numerous applications in industrial and chemical processes like oil and gas processing, annealing of metal and plastic sheets, glass tempering, paper and textile drying petrochemical, premium thermal oil, refining industry, etc. The phenomenon of heat transfer play key role in several chemical engineering phenomena like cooling of chemical equipment, manufacturing of chemical materials, etc. In view of the practical applications, various researchers are engaged to investigate heat transfer phenomenon in various fluids models. The analysis of heat transfer in viscous flow caused by stretching surface was conducted by Gupta and Gupta [1]. Another important study is due to Lawrence and Rao [2] which is concerned with the influence of various parameters on flow of non-Newtonian fluid causes by permeable heated stretching surface. They obtained closed form solution of their problem and claimed that the solution is not unique. Rollins and Vajravelu [3] adopted analytic

\* Corresponding author, e-mail: sk\_iiu@yahoo.com

technique to obtain solution of a problem regarding heat transfer from moving surface immersed in non-Newtonian fluid model. Bhattacharyya [4] implemented shooting technique to highlight the flow of Newtonian fluid caused by shrinking surface. Sahoo [5] discussed the Heimenz flow of third grade fluid over a heated surface. He reported that the velocity at wall is suppressed by increasing third grade fluid parameter. Abbas *et al.* [6] discussed similar solution for Maxwell fluid-flow generated by heated stretching surface. The heat transfer in peristaltic Newtonian flow has been studied by Ali *et al.* [7]. Mahantesh *et al.* [8] examined heat absorption/generation effects in laminar flow of Walters-B fluid model influenced by permeable surface. Oztop *et al.* [9] considered three different cases for mixed convection flow in differentially heated square cavity. Karimi-Fard *et al.* [10] performed numerical computations for double-diffusive natural convection flow in a porous square cavity. According to them, the boundary and inertia effects have significant role in double-diffusive convection. The influence of heat source/sink in flow of micropolar fluid over linearly stretched surface was addressed by Abo-Eldahab and El Aziz [11]. Mandal and Mukhopadhyay [12] presented numerical computations regarding steady-flow of linear fluid model with heat transfer. Malik *et al.* [13] used Sisko fluid model and convective boundary conditions to predict the heat transfer phenomena. Sharma *et al.* [14] presented numerical solution to study heat transfer in non-Newtonian flow caused by exponential stretching surface. Khan *et al.* [15] used Von Karman-Pohlhausen method to solve the integral equations to predict heat transfer phenomenon in viscous fluid-flow over infinite circular cylinder. Gomaa and Taweel [16] analyzed the oscillatory flow of viscous fluid generated by heated vertical flat surface. Further contribution on this topic can be found in [16-20].

The literature survey indicates that aforementioned studies made use of simple Fourier law of heat flux [21] which states that heat flux is proportional to temperature gradient. However, it is observed that this model is applicable to macroscopic systems where time scale of the system is higher than average relaxation time. The mathematical modeling based on this law shows that heat equation results in parabolic form which shows that the whole system is instantly influenced by the initial disturbance. To the best of our knowledge, Cattaneo [22] was first who proposed an extension in the Fourier law by introducing a relaxation time expression and derived a single equation for temperature field. One of the important feature of this law is that it allows the heat transportation via propagation of finite speed thermal waves. The work of [22] was extended by Christov [23] by using Oldroyd upper convective derivative [24]. Pranesh and Kiran [25] introduced Maxwell-Cattaneo heat conduction law to discuss the Rayleigh-Benard magneto convection in a viscoelastic fluid. Straughan [26] presented the analysis of steady-flow of linear fluid model using Cattaneo heat flux modeling presence of thermal relaxation effects. Haddad [27] employed this model to study thermal instability in viscous flow through porous media. Han *et al.* [28] used Cattaneo-Christov expression to study the heat transfer effects in Maxwell fluid over a stretching plate. Hayat *et al.* [29] studied the heat transfer analysis in stagnation-point flow based on Cattaneo-Christov heat flux. Khan *et al.* [30] applied a numerical scheme based on shooting method to evaluate the influence of heat transfer by using this law in flow due to bi-directional stretching surface. Li *et al.* [31] obtained self-similar solution for problem regarding steady-flow of Maxwell fluid and heat transfer by using Cattaneo-Christov heat flux model. Mustafa [32] examined the rotating flow of Maxwell fluid caused by moving surface with the Cattaneo-Christov heat flux model. More recent studies regarding flow of various fluids based on this law can be found in refs. [33-36].

Bearing in mind the previous attempts, as well as the industrial and practical importance of these problems, the main aim of this study is to analyze the unsteady-flow of Maxwell

fluid [37-40] due to oscillatory stretching sheet by using Cattaneo-Christov heat flux model proposed [23] and convective boundary conditions [41-43]. Analytic expressions for both velocity and temperature profiles are obtained using homotopy analysis method (HAM). This study presents MHD flow of non-Newtonian fluid over an oscillatory stretching surface which have immense importance in many industrial and chemical engineering processes. Particularly, the problem presented here with considered geometry has many industrial applications like hot rolling, fibers spinning, manufacturing of rubber sheets. Moreover, the MHD effects are useful in MHD power generating systems, telephone system, computers, X-rays and scanning devices, etc. The formulated problem is solved analytically by HAM. A detailed analysis for several important parameters is presented.

### Flow analysis

Consider 2-D, unsteady-flow of an electrically Maxwell fluid over an oscillatory stretching sheet. The fluid occupies the region  $\bar{y} > 0$ . Let us assumed that sheet is stretched and oscillate periodically along  $\bar{x}$ -axis with velocity  $u = b\bar{x} \sin \omega t$  where  $\omega$  represents the frequency and  $b$  is a constant having the dimension  $[s^{-1}]$ . We have also considered the effects of transverse magnetic field of magnitude  $B_0$  which is imposed normal to the sheet, see fig. 1. Using low Reynolds number assumptions, the effects of induced magnetic field are neglected.

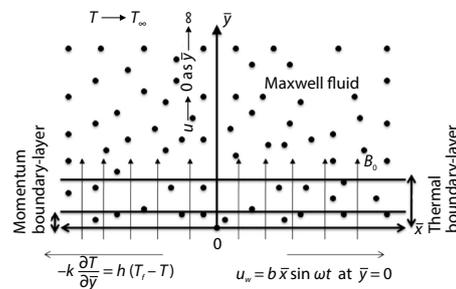


Figure 1. Geometry of problem

The governing boundary-layer equations for 2-D Maxwell fluid-flow are [39]:

$$\frac{\partial u}{\partial \bar{x}} + \frac{\partial v}{\partial \bar{y}} = 0 \quad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial \bar{x}} + v \frac{\partial u}{\partial \bar{y}} = \nu \frac{\partial^2 u}{\partial \bar{y}^2} - \lambda_1 \left[ \frac{\partial^2 u}{\partial t^2} + 2u \frac{\partial^2 u}{\partial \bar{x} \partial t} + 2v \frac{\partial^2 u}{\partial \bar{y} \partial t} + u^2 \frac{\partial^2 u}{\partial \bar{x}^2} + v^2 \frac{\partial^2 u}{\partial \bar{y}^2} + 2uv \frac{\partial^2 u}{\partial \bar{x} \partial \bar{y}} \right] - \frac{\sigma B_0^2}{\rho} \left( u + \lambda_1 v \frac{\partial u}{\partial \bar{y}} + \lambda_1 \frac{\partial u}{\partial t} \right) \quad (2)$$

For the present flow configuration, the initial and boundary conditions are:

$$u = u_w = b\bar{x} \sin \omega t, \quad v = 0, \quad \text{at } \bar{y} = 0, \quad t > 0, \quad u \rightarrow 0, \quad \text{as } \bar{y} \rightarrow \infty \quad (3)$$

In previous equation and denotes the velocity components along and directions, respectively, is the kinematic viscosity, the density, the electric conductivity, and denotes the relaxation parameter.

### Heat transfer analysis

In this section, we are going to formulate heat transfer problem. Unlike typical studies, we derive governing equation of heat transfer using Cattaneo-Christov heat flux model [16]. According to this model the heat flux and temperature gradient are related through following expression:

$$\mathbf{q} + \lambda_2 \left[ \frac{\partial \mathbf{q}}{\partial t} + \mathbf{V} \nabla \mathbf{q} - \mathbf{q} \nabla \mathbf{V} + (\nabla \mathbf{V}) \mathbf{q} \right] = -k \nabla T \quad (4)$$

where  $\mathbf{q}$  represents the heat flux,  $\lambda_2$  – denotes the relaxation time of the heat flux,  $\mathbf{V}$  – the velocity vector,  $T$  – the Maxwell fluid temperature,  $k$  – the thermal conductivity. Equation (4) reduces to well-known Fourier law for  $\lambda_2 = 0$ . For incompressible fluid  $\Delta \mathbf{V}$  and therefore eq. (4) becomes:

$$\mathbf{q} + \lambda_2 \left[ \frac{\partial \mathbf{q}}{\partial t} + \mathbf{V} \nabla \mathbf{q} - \mathbf{q} \nabla \mathbf{V} \right] = -k \nabla T \quad (5)$$

The energy equation for incompressible fluid after neglecting viscous dissipation effects:

$$\rho c_p \left[ \frac{\partial T}{\partial t} + \mathbf{V} \nabla T \right] = -\nabla \mathbf{q} \quad (6)$$

where  $c_p$  is the specific heat. Elimination of  $\mathbf{q}$  from eqs. (5) and (6) yields to the following single equation for the temperature field [16]:

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + \lambda_2 \left[ \begin{aligned} & \frac{\partial^2 T}{\partial t^2} + 2u \frac{\partial^2 T}{\partial x \partial t} + 2v \frac{\partial^2 T}{\partial y \partial t} + \frac{\partial v}{\partial t} \frac{\partial T}{\partial y} + u \frac{\partial u}{\partial x} \frac{\partial T}{\partial x} + v \frac{\partial v}{\partial y} \frac{\partial T}{\partial y} + \\ & + u \frac{\partial v}{\partial x} \frac{\partial T}{\partial y} + v \frac{\partial u}{\partial y} \frac{\partial T}{\partial x} + 2uv \frac{\partial^2 T}{\partial x \partial y} + u^2 \frac{\partial^2 T}{\partial x^2} + v^2 \frac{\partial^2 T}{\partial y^2} \end{aligned} \right] = \alpha \frac{\partial^2 T}{\partial y^2} \quad (7)$$

where  $\alpha = k / \rho c_p$  is thermal diffusivity. Equation (7) is subjected to the convective boundary conditions given:

$$-k \frac{\partial T}{\partial y} = h(T_f - T), \quad \text{at } \bar{y} = 0, \quad t > 0, \quad T \rightarrow T_\infty \text{ as } \bar{y} \rightarrow \infty \quad (8)$$

where  $h$  denotes the heat transfer coefficient. Moreover,  $T_f$  and  $T_\infty$  are the convective fluid temperatures below the surface and ambient fluid temperature, respectively.

### Dimensionless formulation

Before going to the solution of problem, it is better to reduce the number of independent variables in eqs. (2) and (7). We introduce following dimensionless quantities [44]:

$$y = \sqrt{\frac{b}{\nu}} \bar{y}, \quad \tau = t\omega, \quad u = b\bar{x}f_y(y, \tau), \quad v = -\sqrt{\nu b}f(y, \tau) \quad (9)$$

$$\theta(y, \tau) = \frac{T - T_\infty}{T_f - T_\infty} \quad (10)$$

Using eqs. (9) and (10), the continuity equation is identically satisfied, and eqs. (2) and (7), are transformed into following forms:

$$f_{yyy} - \beta \left[ \begin{aligned} & S^2 f_{y\tau\tau} + 2S(f_y f_{y\tau} - ff_{yy\tau}) - \\ & - 2ff_y f_{yy} + f^2 f_{yyy} \end{aligned} \right] - S(1 + \beta M^2) f_{y\tau} - f_y^2 - M^2 f_y + (1 + M^2 \beta) ff_{yy} = 0 \quad (11)$$

$$\frac{1}{\text{Pr}} \theta_{yy} + f\theta_y - S\theta_\tau - \gamma(S^2 \theta_{\tau\tau} - 2Sf\theta_{y\tau} - Sf_\tau \theta_y + ff_y \theta_y + f^2 \theta_{yy}) = 0 \quad (12)$$

Similarly, the boundary conditions of problem under consideration become:

$$f_y(0, \tau) = \sin \tau, \quad f(0, \tau) = 0, \quad \theta_y(0, \tau) = -\gamma_1 [1 - \theta(0, \tau)], \quad f_y(\infty, \tau) = 0, \quad \theta(\infty, \tau) = 0 \quad (13)$$

where  $M = (\sigma B_0^2 / \rho b)^{1/2}$  represents the Hartmann number,  $\beta = \lambda_1 b$  denotes Deborah number,  $\gamma = \lambda_2 b$  – the dimensionless relaxation time of heat flux,  $S = \omega / b$  – the ratio of oscillation frequency to stretching rate,  $\gamma_1 = (h/k)(\nu/b)^{1/2}$  represent the Biot number, and  $\text{Pr} = \nu/\alpha$  – the Prandtl number. Equation (12) represents the energy equation based on Fourier law for  $\gamma = 0$ . For steady-flow, eq. (11) reduces:

$$f_{yyy} - \beta [f^2 f_{yyy} - 2ff_y f_{yy}] - f_y^2 - M^2 f_y + (1 + M^2 \beta) ff_{yy} = 0$$

Moreover, for  $\beta = 0$  it reduces to corresponding equation for flow of hydromagnetic viscous fluid. The corresponding equation for flow of hydrodynamic viscous fluid can be recovered by taking [44].

**Homotopy analysis method**

The dimensionless PDE (11) and (12) are highly non-linear in nature and therefore exact solution is difficult to obtain. Therefore, we implement HAM to compute series solution of these PDE subject to boundary conditions (13). This method was originally proposed by Liao [45] and then successfully applied by many researchers in various disciplines of science and engineering [46-49]. To proceed with the solution, we suggest following initial approximations:

$$f_0(y, \tau) = \sin \tau [1 - \exp(-y)], \quad \theta_0(y) = \frac{\gamma_1 \exp(-y)}{1 + \gamma_1} \tag{14}$$

The auxiliary linear operators are:

$$\mathcal{L}_f(f) = \frac{\partial^3 f}{\partial y^3} - \frac{\partial f}{\partial y}, \quad \mathcal{L}_\theta(f) = \frac{\partial^2 f}{\partial y^2} - f \tag{15}$$

satisfying:

$$\mathcal{L}_f[A_1 + A_2 \exp(y) + A_3 \exp(-y)] = 0 \tag{16}$$

$$\mathcal{L}_\theta[A_4 \exp(y) + A_5 \exp(-y)] = 0 \tag{17}$$

where  $A_i$  (1, 2,...5) represent constants. The zeroth-order deformation problems for give problem:

$$(1 - p)\mathcal{L}_f[\hat{f}(y, \tau; p)] - f_0(y, \tau) = ph_f N_f[\hat{f}(y, \tau; p)] \tag{18}$$

$$(1 - p)\mathcal{L}_\theta[\hat{\theta}(y, \tau; p)] - \theta_0(y, \tau) = ph_\theta N_\theta[\hat{f}(y, \tau; p), \hat{\theta}(y, \tau; p)] \tag{19}$$

$$\hat{f}(y, \tau; p)|_{y=0} = 0, \quad \frac{\partial \hat{f}(y, \tau; p)}{\partial y} \Big|_{y=0} = \sin \tau, \quad \frac{\partial \hat{f}(y, \tau; p)}{\partial y} \Big|_{y=\infty} = 0 \tag{20}$$

$$\frac{\partial \hat{\theta}(0, \tau; p)}{\partial y} = -\gamma_1 (1 - \hat{\theta}(0, \tau; p)), \quad \hat{\theta}(\infty, \tau; p) = 0 \tag{21}$$

The non-linear operators are:

$$N_f[\hat{f}(y, \tau; p)] = \frac{\partial^3 \hat{f}(y, \tau; p)}{\partial y^3} - S(1 + \beta M^2) \frac{\partial^2 \hat{f}(y, \tau; p)}{\partial y \partial \tau} - M^2 \frac{\partial \hat{f}(y, \tau; p)}{\partial y} - \left[ \frac{\partial \hat{f}(y, \tau; p)}{\partial y} \right]^2 + (1 + \beta M^2) \hat{f}(y, \tau; p) \frac{\partial^2 \hat{f}(y, \tau; p)}{\partial y^2} - \left. \begin{aligned} & -\beta \left\{ S^2 \frac{\partial^3 \hat{f}(y, \tau; p)}{\partial y \partial \tau^2} + 2S \left[ \frac{\partial \hat{f}(y, \tau; p)}{\partial y} \frac{\partial^2 \hat{f}(y, \tau; p)}{\partial \tau \partial y} - \hat{f}(y, \tau; p) \frac{\partial^3 \hat{f}(y, \tau; p)}{\partial \tau \partial y^2} \right] + \right. \\ & \left. + \hat{f}^2(y, \tau; p) \frac{\partial^3 \hat{f}(y, \tau; p)}{\partial y^3} - 2\hat{f}(y, \tau; p) \frac{\partial \hat{f}(y, \tau; p)}{\partial y} \frac{\partial^2 \hat{f}(y, \tau; p)}{\partial y^2} \right\} \end{aligned} \right] \tag{22}$$

$$N_\theta[\hat{\theta}(y, \tau; p), \hat{f}(y, \tau; p)] = \frac{1}{\text{Pr}} \frac{\partial^2 \hat{\theta}(y, \tau; p)}{\partial y^2} + \hat{f}(y, \tau; p) \frac{\partial \hat{\theta}(y, \tau; p)}{\partial y} - S \frac{\partial \hat{\theta}(y, \tau; p)}{\partial \tau} -$$

$$- \gamma \left[ S^2 \frac{\partial^2 \hat{\theta}(y, \tau; p)}{\partial \tau^2} - 2S \hat{f}(y, \tau; p) \frac{\partial^2 \hat{\theta}(y, \tau; p)}{\partial y \partial \tau} - S \frac{\partial \hat{f}(y, \tau; p)}{\partial \tau} \frac{\partial \hat{\theta}(y, \tau; p)}{\partial y} + \right. \quad (23)$$

$$\left. + \hat{f}(y, \tau; p) \frac{\partial \hat{f}(y, \tau; p)}{\partial y} \frac{\partial \hat{\theta}(y, \tau; p)}{\partial y} + \hat{f}^2(y, \tau; p) \frac{\partial^2 \hat{\theta}(y, \tau; p)}{\partial y^2} \right]$$

The solution of zeroth-order deformation problems at  $p = 0$  and  $p = 1$  is:

$$\hat{f}(y, \tau; 0) = f_0(y, \tau), \hat{f}(y, \tau; 1) = f(y, \tau) \quad (24)$$

$$\hat{\theta}(y, \tau; 0) = \theta_0(y, \tau), \hat{\theta}(y, \tau; 1) = \theta(y, \tau) \quad (25)$$

Using Taylor's series expansion, we have:

$$\hat{f}(y, \tau; p) = f_0(y, \tau) + \sum_{m=1}^{\infty} f_m(y, \tau) p^m, \quad f_m(y, \tau) = \frac{1}{m!} \frac{\partial^m \hat{f}(y, \tau; p)}{\partial p^m} \quad (26)$$

$$\hat{\theta}(y, \tau; p) = \theta_0(y, \tau) + \sum_{m=1}^{\infty} \theta_m(y, \tau) p^m, \quad \theta_m(y, \tau) = \frac{1}{m!} \frac{\partial^m \hat{\theta}(y, \tau; p)}{\partial p^m} \quad (27)$$

The convergence of HAM depends upon  $h_f$  and  $h_\theta$ . We assume  $h_f$  that  $h_\theta$  and are selected so that series solution converges at  $p = 1$ . Therefore:

$$f(y, \tau; p) = f_0(y, \tau) + \sum_{m=1}^{\infty} f_m(y, \tau) \quad (28)$$

$$\theta(y, \tau) = \theta_0(y, \tau) + \sum_{m=1}^{\infty} \theta_m(y, \tau) \quad (29)$$

where  $f_m$  and  $\theta_m$  can be computed through the  $m$ th-order of deformation problem given by:

$$\mathcal{L}_f [f_m(y, \tau) - \chi_m f_{m-1}(y, \tau)] = h_f R_m^f(y, \tau) \quad (30)$$

$$\mathcal{L}_\theta [\theta_m(y, \tau) - \chi_m \theta_{m-1}(y, \tau)] = h_\theta R_m^\theta(y, \tau) \quad (31)$$

$$f_m(0, \tau) = 0, \quad \frac{\partial f_m(0, \tau)}{\partial y} = 0, \quad \frac{\partial f_m(\infty, \tau)}{\partial y} = 0 \quad (32)$$

$$\frac{\partial \theta_m(0, \tau)}{\partial y} - \gamma_1 \theta_m(0, \tau) = \theta_m(\infty, \tau) = 0 \quad (33)$$

$$R_m^f(y, \tau) = \frac{\partial^3 f_{m-1}}{\partial y^3} - S(1 + \beta M^2) \frac{\partial^2 f_{m-1}}{\partial y \partial \tau} - M^2 \frac{\partial f_{m-1}}{\partial y} -$$

$$- \beta S^2 \frac{\partial^3 f_{m-1}}{\partial y \partial \tau^2} + (1 + M^2 \beta) \sum_{k=0}^{m-1} f_{m-1-k} \frac{\partial^2 f_k}{\partial y^2} - \sum_{k=0}^{m-1} \left( \frac{\partial f_{m-1-k}}{\partial y} \frac{\partial f_k}{\partial y} \right) -$$

$$- \beta \sum_{k=0}^{m-1} \left\{ \left( 2S \frac{\partial f_{m-1-k}}{\partial y} \frac{\partial^2 f_k}{\partial y \partial \tau} - 2S f_{m-1-k} \frac{\partial^3 f_k}{\partial \tau \partial y^2} \right) - f_{m-1-k} \sum_{l=0}^k 2 \frac{\partial f_{k-l}}{\partial y} \frac{\partial^2 f_l}{\partial y^2} - f_{k-l} \frac{\partial^3 f_l}{\partial y^3} \right\} \quad (34)$$

$$R_m^f(y, \tau) = \frac{1}{Pr} \frac{\partial^2 \theta_{m-1}}{\partial y^2} - S \frac{\partial \theta_{m-1}}{\partial \tau} - \gamma S^2 \frac{\partial^2 \theta_{m-1}}{\partial \tau^2} + \sum_{k=0}^{m-1} f_{m-1-k} \frac{\partial \theta_k}{\partial y} + S \gamma \sum_{k=0}^{m-1} \left( 2 f_{m-1-k} \frac{\partial^2 \theta_k}{\partial y \partial \tau} + \frac{\partial f_{m-1-k}}{\partial \tau} \frac{\partial \theta_k}{\partial y} \right) - \gamma \sum_{k=0}^{m-1} \left\{ f_{m-1-k} \sum_{l=0}^k \left( \frac{\partial f_{k-l}}{\partial y} \frac{\partial \theta_l}{\partial y} + f_{k-l} \frac{\partial^2 \theta_l}{\partial y^2} \right) \right\} \quad (35)$$

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \quad (36)$$

The general solution is of the form:

$$f_m(y, \tau) = f_m^*(y, \tau) + A_1 + A_2 \exp(y) + A_3 \exp(-y) \quad (37)$$

$$\theta_m(y, \tau) = \theta_m^*(y, \tau) + A_4 \exp(y) + A_5 \exp(-y) \quad (38)$$

where  $f_m^*(y, \tau)$  and  $\theta_m^*(y, \tau)$  represents the particular solution. Using eqs. (20) and (21), the constants  $A_i (i = 1, 2, \dots, 5)$  are eliminated as:

$$A_2 = A_4 = 0, \quad A_3 = \frac{\partial f_m^*(0, \tau)}{\partial y}, \quad A_1 = -A_3 - f_m^*(0, \tau), \quad A_5 = \frac{1}{\gamma_1 + 1} \left[ \frac{\partial \theta_m^*(0, \tau)}{\partial y} - \gamma_1 \theta_m^*(0, \tau) \right] \quad (39)$$

### Discussion

The auxiliary parameters involved in the analytic expressions are  $h_f$  and  $h_\theta$ . The convergence region as well as rate of approximations can be estimated for solutions by these parameters. The plots of  $f_{yy}(0, \tau)$  vs.  $h_f$  and  $\theta_y(0, \tau)$  vs.  $h_\theta$  have been shown in fig. 2 at 6<sup>th</sup> order of approximations to estimate suitable range of these parameter for convergent solutions. It is observed that for chosen set of involved parameters the admissible ranges for  $h_f$  and  $h_\theta$  are:  $-2 \leq h_f \leq -0.3$ ,  $-1 \leq h_\theta \leq 0$ , respectively. The plots of residual error for  $f$  and for a particular set of involved parameters at 6<sup>th</sup> order of approximation is shown in fig. 3(a) when  $h_f = -0.6$ . Clearly, the maximum error over the whole domain is less than  $6 \cdot 10^{-6}$ . Similarly, fig. 3(b) testifies that residual error for  $\theta$  is in acceptable range when  $h_f = -6$  and  $h_\theta = -0.8$ .

Now we come to the discussion of graphical results concerning the velocity and temperature distribution for diverse values of various flow parameters like Hartmann number, Deborah number,  $\beta$ , relaxation time of the heat flux  $\gamma$ , Biot number and Prandtl number.

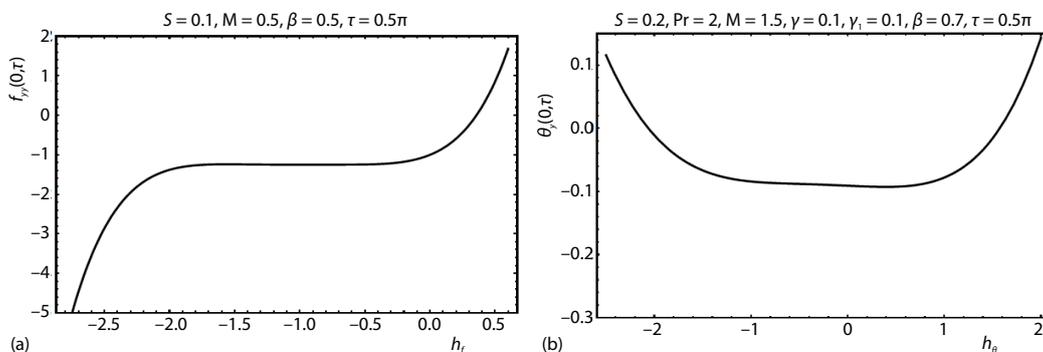


Figure 2. The  $h$ -curves for (a) velocity, (b) temperature profiles

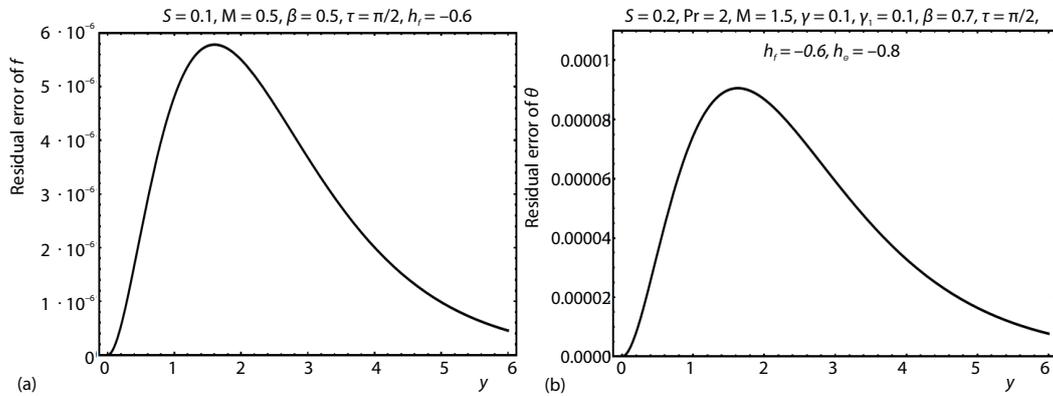


Figure 3. Residual error for (a) velocity, (b) temperature profiles

Figure 4(a) explains the effects of Deborah number on the transverse velocity component  $f'$  by keeping  $S = 0.2$ ,  $M = 0.5$ , and  $\tau = \pi/2$ . This figure shows that velocity decreases by increasing Deborah number. For viscous fluid  $\beta = 0$  the momentum boundary-layer is thicker as compared to non-Newtonian fluid. From physical and experimental point of view it is seen that at lower values of Deborah number, the fluid behaves much like liquid whereas the fluid shows viscoelastic solid like behavior at high Deborah number due to an increase in viscous properties and thus the fluid velocity starts to decelerate which is noted in fig. 4(a). Moreover, the thickness of the boundary-layer is suppressed for higher values of Deborah number.

The transverse distributions of the flow velocity  $f'$  for specific values Hartmann number is shown in fig. 4(b). Application of strong magnetic force tends to resist the velocity of fluid particles near the surface. This is in fact due to the fact that the presence of magnetic force produces the Lorentz force which resists the flow produced by oscillating sheet.

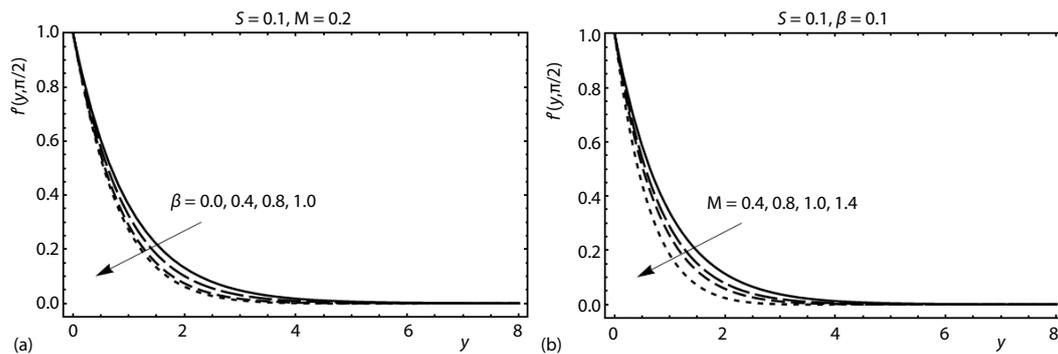


Figure 4. The velocity profile for different values of (a)  $\beta$ , (b)  $M$

The velocity  $f'$  as function of time at a specific location  $y = 0.25$  is plotted for different values of Deborah number and Hartmann number in fig. 5. A decrease in amplitude of flow velocity is noted with an increase in the Deborah number, fig. 5(a). In fact for larger Deborah number, the viscous forces are dominant which restrict the motion of fluids particles and as a result amplitude is decreased. Figure 5(b) is sketched to examine the behavior of Hartmann number on  $f'$ . Here, it is observed that the amplitude of the velocity,  $f'$ , decreases with increasing

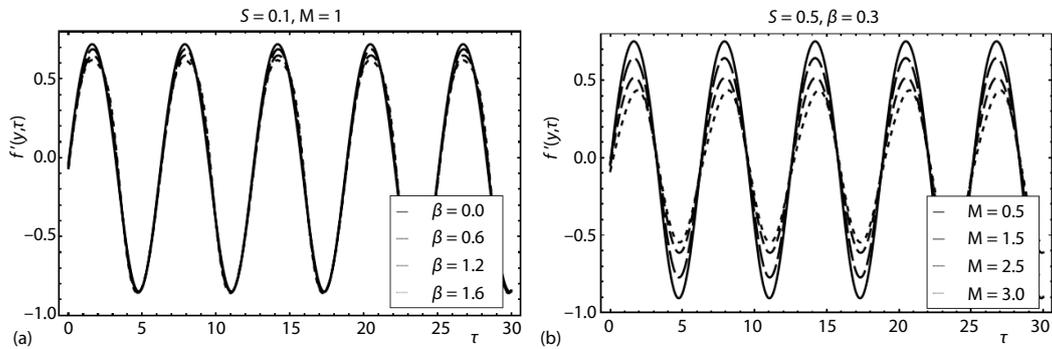


Figure 5. Variation of velocity with time (a) effects of  $\beta$ , (b) effects of  $M$

the values of Hartmann number. Again, this suppression in the amplitude is due to the resistive force produced due to application of magnetic field normal to the sheet.

The dependence of the fluid temperature on the Prandtl number and  $\gamma$  is sketched in fig. 6. Increasing the Prandtl number results in the thickening of thermal boundary-layer. It can be justified because thermal diffusivity decreases for large values of Prandtl number which results in decrease of temperature and corresponding thermal boundary-layer thickness. However, for non-zero values of  $\gamma$  the thermal boundary-layer thickness decreases more rapidly with increasing Prandtl number.

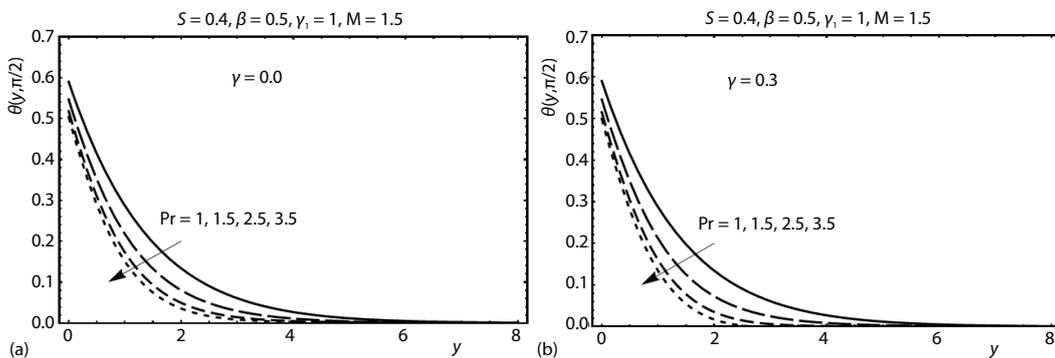


Figure 6. Effects of  $Pr$  on  $\theta$

In fig. 7, we give the variation of temperature field  $\theta$  for various values of  $\gamma$  and for two different values of Prandtl number. It is interesting to note that the temperature and thermal boundary-layer thickness decreases with increasing  $\gamma$ . Further, it is observed that this decrease is faster for larger values of Prandtl number. In figs. 8 and 9, the effects of Deborah number and Hartmann number on temperature field are shown, respectively. Both the parameters effect the temperature field in a similar manner *i. e.* the temperature field increases by increasing of either Deborah and Hartman numbers.

The variation of temperature profile for four different values of thermal Biot number,  $\gamma_1 = 1, 1.5, 2.5, 3.5$  are displayed in fig. 10. Thermal Biot number is associated with heat transfer coefficient,  $h$ , therefore its higher values represent the case of enhanced heat transfer from stretching sheet to the fluid stream. This enhancement in heat transfer is responsible for increase in the temperature of fluid. Figure 11 reflects the influence of ratio of oscillation frequency to stretching rate,  $S$ , on temperature profile,  $\theta$ . One can easily observe that temperature field  $\theta$  is decreased by increasing  $S$ .

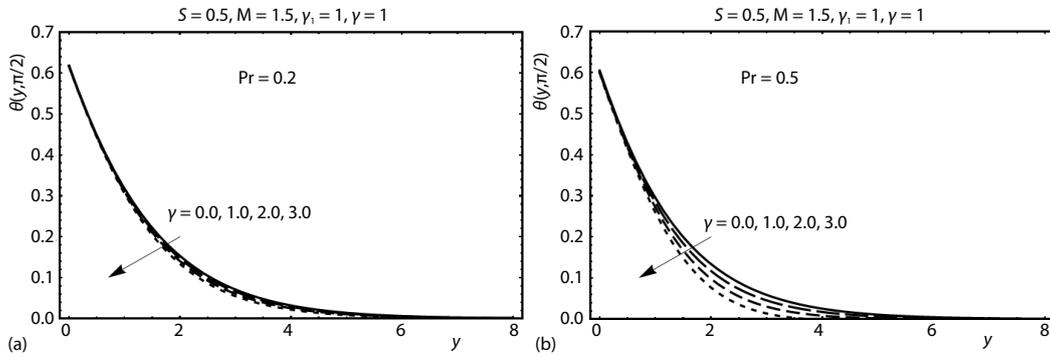


Figure 7. Effects of  $\gamma$  on  $\theta$

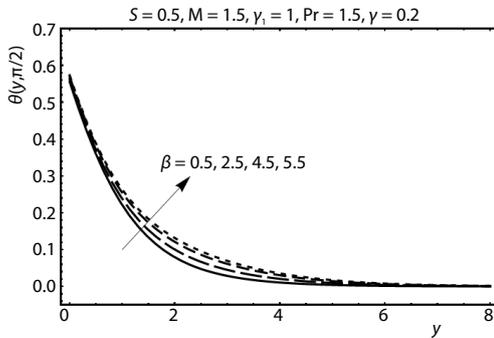


Figure 8. Effects of  $\beta$  on  $\theta$

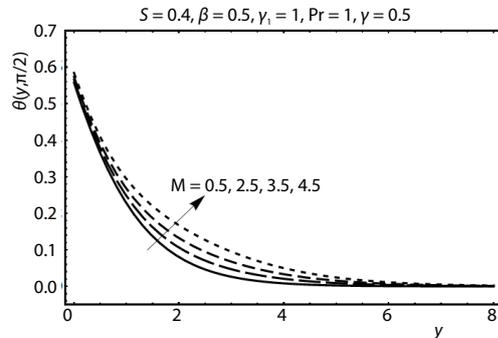


Figure 9. Effects of  $M$  on  $\theta$

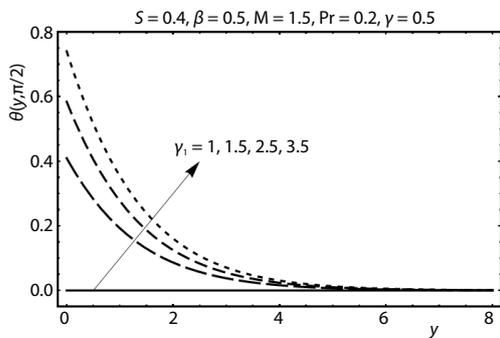


Figure 10. Effects of  $\gamma_1$  on  $\theta$

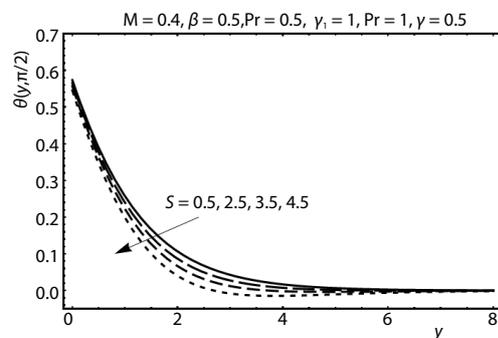


Figure 11. Effects of  $S$  on  $\theta$

In tab. 1, the obtained solution is validated against existing results of Zheng *et al.* [19] and Ali *et al.* [20]. An excellent agreement between both solutions is observed.

**Concluding remarks**

The heat transfer analysis in unsteady-flow of Maxwell fluid by using Cattaneo-Christov heat flux model is presented when plate is stretched periodically. After computing the solution by HAM, a comprehensive analysis has been presented to highlight the effects of various flow parameters. The main findings of the analysis can be summarized as following.

- By increasing Deborah number the fluid velocity is suppressed in the vicinity of the surface in given domain. The same is true with increasing Hartmann number.

**Table 1. Comparison of  $f''(0, \tau)$  for various values of  $\tau$  when  $S = 1$ ,  $M = 12$  in case of Newtonian fluid,  $\beta = 0$**

$\tau$	Zheng et al. [19]	Ali et al. [20]		Present results
		HAM results	Numerical result	
$1.5 \pi$	11.678656	11.678656	11.678656	11.678656
$1.5 \pi$	11.678706	11.678706	11.678707	11.678706
$1.5 \pi$	11.678656	11.678656	11.678656	11.678656

- A oscillations in flow velocity at a specific location are suppressed for larger values of Deborah number and Hartmann number.
- It is noted that the heat transfer rate from sheet to the fluid become slow for larger values of Prandtl number and relaxation time of heat flux.
- The Cattaneo-Christov heat flux model predicts lower values of temperature inside the thermal boundary-layer as compared heat flux model based on Fourier law.
- The rate of heat transfer enhanced by increasing Biot number. Moreover, there is no heat transfer when  $\gamma_1 = 1$ .

## References

- [1] Gupta, P. S., Gupta, A. S., Heat and Mass Transfer on a Stretching Sheet with Suction and Blowing, *The Canadian Journal of Chemical Engineering*, 55 (1977), 6, pp. 744-746
- [2] Lawrence, P. S., Rao, B. N., Heat Transfer in the Flow of a Viscoelastic Fluid over a Stretching Sheet, *Acta Mechanica*, 93 (1992), 3, pp. 53-61
- [3] Rollins, D., Vajravelue, K., Heat Transfer in a Second-Order Fluid over a Continuous Stretching surface, *Acta Mechanica*, 89 (1991), 1-4, pp. 167-178
- [4] Bhattacharyya, K., Boundary-Layer Flow and Heat Transfer over an Exponentially Shrinking Sheet, *Chin. Phys. Lett.*, 28 (2011), 7, 074701
- [5] Sahoo, B., Hiemenz Flow and Heat Transfer of a Third Grade Fluid, *Communications in Non-Linear, Science and Numerical Simulation*, 14 (2009), 3, pp. 811-826
- [6] Abbas, Z., et al., Flow and Heat Transfer of Maxwell Fluid over an Exponentially Stretching Sheet: A Non-Similar Solution, *Heat Transfer Asian Research*, 43 (2014), 3, pp. 233-242
- [7] Ali, N., et al., Heat Transfer Analysis for Peristaltic Flow in a Curved Channel, *International Journal of Heat and Mass Transfer*, 53 (2010), 15-16, pp. 3319-3325
- [8] Mahantesh, M., et al., Heat Transfer in MHD Viscoelastic Boundary-Layer Flow over a Stretching Sheet with Thermal Radiation and Non-Uniform Heat Source/Sink, *Communications in Nonlinear Science and Numerical Simulation*, 16 (2011), 9, pp. 3578-3590
- [9] Osorio, A., et al., On the Natural Convection of Water Near Its Density Inversion in an Inclined Square Cavity, *International Journal of Heat and Mass Transfer*, 47 (2004), 19-20, pp. 4491-4495
- [10] Karimi-Fard, M., et al., Non-Darcian Effects on Double-Diffusive Convection within a Porous Medium, *Numerical Heat Transfer, Part A, Applications*, 31 (1997), 8, pp. 837-852
- [11] Abo-Eldahab, E. M., El Aziz, M. A., Flow and Heat Transfer in a Micropolar Fluid Past a Stretching Surface Embedded in a Non-Darcian Porous Medium with Uniform Free Stream, *Applied Mathematics and Computation*, 162 (2005), 2, pp. 881-899
- [12] Mandal, I. C., Mukhopadhyay, S., Heat Transfer Analysis for Fluid-Flow over an Exponentially Stretching Porous Sheet with Surface Heat Flux in Porous Medium, *Ain Shams Engineering Journal*, 4 (2013), 1, pp. 103-110
- [13] Malik, R., et al., Flow and Heat Transfer in Sisko Fluid with Convective Boundary Condition, *Plos One*, 9 (2014), 10, e107989
- [14] Sharma, P. R., et al., Numerical Solution of Flow and Heat Transfer of a Non-Newtonian Fluid over a Stretching Sheet, *Modelling, Measurement and Control B*, 74 (2005), 1, pp. 45-62
- [15] Khan, W. A., et al., Fluid-Flow Around and Heat Transfer from an Infinite Circular cylinder, *Journal of Heat Transfer*, 127 (2005), 7, pp. 785-790

- [16] Gomaa, H., Taweel, A. M. Al, Effect of Oscillatory Motion on Heat Transfer at Vertical Flat Surfaces, *International Journal of Heat and Mass Transfer*, 48 (2005), 8, pp. 1494-1504
- [17] Su, X. H., et al., MHD Mixed Convective Heat Transfer over a Permeable Stretching Wedge with Thermal Radiation and Ohmic Heating, *Chemical Engineering Science*, 78 (2012), Aug., pp. 1-8
- [18] Rosca, A. V., Pop, I., Flow and Heat Transfer over a Vertical Permeable Stretching/Shrinking Sheet with a Second Order Slip, *International Journal of Heat and Mass Transfer*, 60 (2013), May, pp. 355-364
- [19] Zheng, L. C., et al., Unsteady Heat and Mass Transfer in MHD Flow over an Oscillatory Stretching Surface with Soret and Dufour Effects, *Acta Mechanica Sinica*, 29 (2013), 5, pp. 667-675
- [20] Ali, N., et al., Hydromagnetic Flow and Heat Transfer of a Jeffrey Fluid over an Oscillatory Stretching Surface, *Zeitschrift für Naturforschung A*, 70 (2015), 7a, pp. 567-576
- [21] Fourier, J. B. J., *Théorie Analytique De La Chaleur*, Cosimo, Inc., Paris, 1822
- [22] Cattaneo, C., Sulla Conduzione del calore, (About Heat Conduction in Italian), *AttiSemin. Mat. Fis. Univ. Modena Reggio Emilia*, 3 (1948), 3, pp. 83-101
- [23] Christov, C. I., On Frame Indifferent Formulation of the Maxwell-Cattaneo Model of Finite-Speed Heat Conduction, *Mechanics Research Communications*, 36 (2009), 4, pp. 48-486
- [24] Oldroyd, J. G., On the Formulation of Rheological Equations of State, *Proceedings of the Royal Society A*, 200 (1949), 1063, pp. 523-541
- [25] Pranesh, S. Kiran, R. V., Study of Rayleigh-Benard Magneto Convection in a Micropolar Fluid with Maxwell-Cattaneo Law, *Applied Mathematics*, 1 (2010), 6, pp. 470-480
- [26] Straughan, B., Thermal Convection with the Cattaneo-Christov Model, *International Journal of Heat and Mass Transfer*, 53 (2010), 1-3, pp. 95-98
- [27] Haddad, S. A. M., Thermal Instability in Brinkman Porous Media with Cattaneo-Christov Heat Flux, *International Journal of Heat and Mass Transfer*, 68 (2014), Jan., pp. 659-668
- [28] Han, S., et al., Coupled Flow and Heat Transfer in Viscoelastic Fluid with Cattaneo-Christov Heat Flux NModel, *Applied Mathematics Letters*, 38 (2014), Dec., pp. 87-93
- [29] Hayat, T. et al., Stagnation Point Flow with Cattaneo-Christov Heat Flux and Homogeneous-Heterogeneous Reactions, *Journal of Molecular Liquids*, 220 (2016), Aug., pp. 49-55
- [30] Khan, M., et al., A 3-D Sisko Fluid-flow with Cattaneo-Christov Heat Flux Model and Heterogeneous-Homogeneous Reactions: A numerical study, *Journal of Molecular Liquids*, 238 (2017), July, pp. 19-26
- [31] Li, J., et al., The MHD Viscoelastic Flow and Heat Transfer over a Vertical Stretching Sheet with Cattaneo-Christov Heat Flux Effects, *Journal of Molecular Liquids*, 221 (2016), Sept., pp. 19-25
- [32] Mustafa, M., Cattaneo-Christov Heat Flux Model for Rotating Flow and Heat Transfer of Upper Convected Maxwell Fluid, *AIP Advances*, 5 (2015), 4, 047109
- [33] Abbasi, F. M., et al., Analytical Study of Cattaneo-Christov Heat Flux Model for Boundary-Layer Flow of an Oldroyd-B Fluid, *Chinese Physics B*, 25 (2016), 1, 014701
- [34] Waqas, M., et al., Cattaneo-Christov Heat Flux Model for Flow of Variable Thermal Conductivity Generalized Burgers Fluid, *Journal of Molecular Liquids*, 220 (2016), Aug., pp. 642-648
- [35] Shehzad, S. A., et al., Cattaneo-Christov Heat Flux Model for Third-Grade Fluid-flow towards Exponentially Stretching Sheet, *Applied Mathematics and Mechanics*, 37 (2016), 6, pp. 761-768
- [36] Abbasi, F. M., et al., Influence of Cattaneo-Christov Heat Flux in Flow of an Oldroyd-B Fluid with Variable Thermal Conductivity, *International Journal of Numerical Methods for Heat & Fluid-flow*, 26 (2016), 7, pp. 2271-2282
- [37] Hayat, T., et al., Mass Transfer Effects on the Unsteady-flow of UCM Fluid over a Stretching Sheet, *International Journal of Modern Physics B*, 25 (2011), 21, pp. 2863-78
- [38] Pahlavan, A. A., et al., The MHD Flow of UCM Fluids above Stretching Sheet Using Two-Auxiliary-Parameter Homotopy Analysis Method, *Communication in Nonlinear Science and Numerical Simulation*, 14 (2009), 2, pp. 473-488
- [39] Awais, M. M., et al., Time-Dependent Three-Dimensional Boundary-Layer Flow of a Maxwell Fluid, *Computers & Fluids*, 91 (2014), Mar., pp. 21-27
- [40] Mushtaq, A., et al., Effects of Thermal Radiation on the Stagnation-Point Flow of Upper-Convected Maxwell Fluid over a Stretching Sheet, *Journal of Aerospace Engineering*, 27 (2014), 4, 04014015
- [41] Hayat, T., et al., Steady-flow of Maxwell Fluid with Convective Boundary Conditions, *Zeitschrift für Naturforschung A*, 66a (2011), 6-7, pp.417-422
- [42] Hayat, T., et al., Steady-flow of an Eyring Powell Fluid over a Moving Surface with Convective Boundary Conditions, *International Journal of Heat and Mass Transfer*, 55 (2012), 7-8, pp. 1817-1822

- [43] Hayat, T., On Mixed Convection Stagnation Point Flow of Second Grade Fluid with Convective Boundary Condition and Thermal Radiation, *International Journal of Nonlinear Science and Numerical Simulation*, 15 (2014), 1, pp. 27-34
- [44] Wang, C. Y., Non-Linear Streaming Due to the Oscillatory Stretching of a Sheet in a Viscous, *Acta Mechanica*, 72 (1988), 3-4, pp. 261-268
- [45] Liao, S. J., The Proposed Homotopy Analysis Technique for the Solution of Non-Linear Problems, Ph. D. thesis, Shanghai Jiao Tong University, Shanghai, China, 1992
- [46] Abbasbandy, S., Homotopy Analysis Method for Heat Radiation Equations, *International Communications in Heat and Mass Transfer*, 34 (2007), 3, pp. 380-387
- [47] Turkyilmazoglu, M., Analytic Approximate Solutions of Rotating Disk Boundary-Layer Flow Subject to a Uniform Suction or Injection, *International Journal of Mechanical Sciences*, 52 (2010), 12, pp. 1735-1744
- [48] Turkyilmazoglu, M., A Note on the Homotopy Analysis Method, *Applied Mathematics Letters*, 23 (2010), 10, pp. 1226-1230
- [49] Liao, S. J., *Homotopy Analysis Method in Non-Linear Differential Equations*, Higher Education Press, Beijing, 2012