

HEAT TRANSFER ANALYSIS BASED ON CATTANEO–CHRISTOV HEAT FLUX MODEL AND CONVECTIVE BOUNDARY CONDITIONS FOR FLOW OVER AN OSCILLATORY STRETCHING SURFACE

by

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Abstract: *In this study, we investigate the heat transfer characteristics in unsteady boundary layer flow of Maxwell fluid by using Cattaneo-Christov heat flux model and convective boundary conditions. The flow is caused by a sheet which is stretched periodically back and forth in its own plane. The physical model that takes into account the effects of constant applied magnetic field is transformed into highly nonlinear partial differential equations under boundary layer approximations. The solution of dimensionless version of these equations is developed using homotopy analysis method. The simulations are presented in the form of temperature and velocity profiles for suitable range of physical parameters. The obtained results illustrate that an increase in Deborah number and Hartmann number suppress the velocity profiles. It is further observed that Cattaneo-Christov heat flux model predicts the suppression of thermal boundary layer thickness as compared to Fourier law.*

Keywords: *Maxwell fluid, Cattaneo-Christov heat flux model, oscillatory stretching sheet, homotopy analysis method.*

1. Introduction

The study of convective of heat transfer gained great attention of investigators because of its numerous applications in industrial and chemical processes like oil and gas processing, annealing of metal and plastic sheets, glass tempering, paper and textile drying petrochemical, premium thermal oil, refining industry etc. The phenomenon of heat transfer play key role in several chemical engineering phenomena like cooling of chemical equipment, manufacturing of chemical materials etc. In view of the practical applications, various researchers are engaged to investigate heat transfer phenomenon in various fluids models. The analysis of heat transfer in viscous flow caused by stretching surface was conducted by Gupta [1]. Another important study is due to Lawrence and Rao [2] which is concerned with the influence of various parameters on flow of non-Newtonian fluid causes by permeable heated stretching surface. They obtained closed form solution of their problem and claimed that the solution is not unique. Rollins and Vajravelu [3] adopted analytic technique to obtained solution of a problem regarding heat transfer

from moving surface immersed in non-Newtonian fluid model. Bhattacharyya [4] implemented shooting technique to highlight the flow of Newtonian fluid caused by shrinking surface. Sahoo[5] discussed the Heimenz flow of third grade fluid over a heated surface. He reported that the velocity at wall is suppressed by increasing third grade fluid parameter. Abbas et al. [6] discussed similar solution for Maxwell fluid flow generated by heated stretching surface. The heat transfer in peristaltic Newtonian flow has been studied by Ali et al. [7]. Mahantesh et al. [8] examined heat absorption/generation effects in laminar flow of Walters-B fluid model influenced by permeable surface. Oztop and Dagtekin [9] considered three different cases for mixed convection flow in differentially heated square cavity. Karimi-Fard et al. [10] performed numerical computations for double-diffusive natural convection flow in a porous square cavity. According to them, the boundary and inertia effects have significant role in double-diffusive convection. The influence of heat source/sink in flow of micropolar fluid over linearly stretched surface was addressed by Eldaha and Aziz [11]. Mandal and Mukhopadhyay [12] presented numerical computations regarding steady flow of linear fluid model with heat transfer. Malik et al. [13] used Sisko fluid model and convective boundary conditions to predict the heat transfer phenomena. Sharma et al. [14] presented numerical solution to study heat transfer in non-Newtonian flow caused by exponential stretching surface. Khan et al. [15] used Von Karman–Pohlhausen method to solve the integral equations to predict heat transfer phenomenon in viscous fluid flow over infinite circular cylinder. Goma and Taweel [16] analyzed the oscillatory flow of viscous fluid generated by heated vertical flat surface. Further contribution on this topic can be found in [16-20].

The literature survey indicates that aforementioned studies made use of simple Fourier law of heat flux [21] which states that heat flux is proportional to temperature gradient. However, it is observed that this model is applicable to macroscopic systems where time scale of the system is higher than average relaxation time. The mathematical modeling based on this law shows that heat equation results in parabolic form which shows that the whole system is instantly influenced by the initial disturbance. To the best of our knowledge, Cattaneo [22] was first who proposed an extension in the Fourier law by introducing a relaxation time expression and derived a single equation for temperature field. One of the important feature of this law is that it allows the heat transportation via propagation of finite speed thermal waves. The work of [22] was extended by Christov [23] by using Oldroyd upper convective derivative [24]. Subbarama et al. [25] introduced Maxwell-Cattaneo heat conduction law to discuss the Rayleigh-Bénard magneto convection in a viscoelastic fluid. Straughan [26] presented the analysis of steady flow of linear fluid model using Cattaneo heat flux model in presence of thermal relaxation effects. Haddad [27] employed this model to study thermal instability in viscous flow through porous media. Han et al. [28] used Cattaneo-Christov expression to study the heat transfer effects in Maxwell fluid over a stretching plate. Hayat et al. [29] studied the heat transfer analysis in stagnation-point flow based on Cattaneo-Christov heat flux. Khan et al. [30] applied a numerical scheme based on shooting method to evaluate the influence of heat transfer by using this law in flow due to bi-directional stretching surface. Li et al. [31] obtained self-similar solution for problem regarding steady flow of Maxwell fluid and heat transfer by using Cattaneo-Christov heat flux model. Mustafa [32] examined the rotating flow of Maxwell fluid caused by moving surface with the Cattaneo-Christov heat flux model. More recent studies regarding flow of various fluids based on this law can be found in refs. [33-36].

Bearing in mind the previous attempts, as well as the industrial and practical importance of these problems, the main aim of this study is to analyze the unsteady flow of Maxwell fluid [37-40] due to oscillatory stretching sheet by using Cattaneo-Christov heat flux model proposed in [23] and convective boundary conditions [41-43]. Analytic expressions for both velocity and temperature profiles are obtained using homotopy analysis method. This study presents MHD flow of non-Newtonian fluid over an oscillatory stretching surface which have immense importance in many industrial and chemical engineering processes. Particularly, the problem presented here with considered geometry has many industrial applications like hot rolling, fibers spinning, manufacturing of rubber sheets. Moreover, themagnetohydrodynamic (MHD) effects are useful in MHD power generating systems, telephone system, computers, X-rays and scanning devices etc. The formulated problem is solved analytically by homotopy analysis method. A detailed analysis for several important parameters is presented.

2. Flow Analysis

Consider two-dimensional, unsteady flow of an electrically Maxwell fluid over an oscillatory stretching sheet. The fluid occupies the region $\bar{y} > 0$. Let us assumed that sheet is stretched and oscillate periodically along \bar{x} -axis with velocity $u = b\bar{x} \sin \omega t$ where ω represents the frequency and b is a constant having the dimension $[T]^{-1}$. We have also considered the effects of transverse magnetic field of magnitude B_0 which is imposed normal to the sheet (see Fig. 1). Using low Reynolds number assumptions, the effects of induced magnetic field are neglected. The governing boundary layer equations for two-dimensional Maxwell fluid flow are [39]

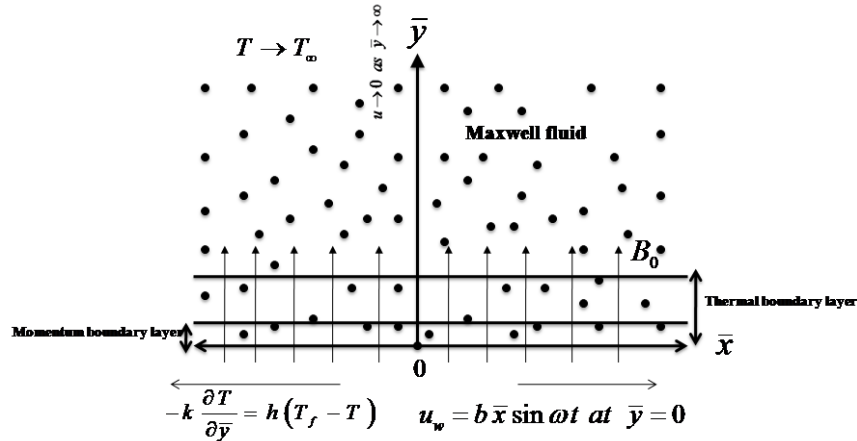


Fig. 1: Geometry of problem

$$\frac{\partial u}{\partial \bar{x}} + \frac{\partial v}{\partial \bar{y}} = 0, \quad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial \bar{x}} + v \frac{\partial v}{\partial \bar{y}} = \nu \frac{\partial^2 u}{\partial \bar{y}^2} - \lambda_1 \left[\begin{array}{l} \frac{\partial^2 u}{\partial t^2} + 2u \frac{\partial^2 u}{\partial \bar{x} \partial t} + 2v \frac{\partial^2 u}{\partial \bar{y} \partial t} \\ + u^2 \frac{\partial^2 u}{\partial \bar{x}^2} + v^2 \frac{\partial^2 u}{\partial \bar{y}^2} + 2uv \frac{\partial^2 u}{\partial \bar{x} \partial \bar{y}} \end{array} \right] - \frac{\sigma B_0^2}{\rho} \left(u + \lambda_1 v \frac{\partial u}{\partial \bar{y}} + \lambda_1 \frac{\partial u}{\partial t} \right), \quad (2)$$

For the present flow configuration, the initial and boundary conditions are

$$u = u_\omega = b\bar{x} \sin \omega t, \quad v = 0, \quad \text{at} \quad \bar{y} = 0, \quad t > 0, \quad u \rightarrow 0, \quad \text{as} \quad \bar{y} \rightarrow \infty, \quad (3)$$

In above equation u and v denotes the velocity components along \bar{x} - and \bar{y} -directions, respectively, ν is the kinematic viscosity, ρ is the density, σ is the electric conductivity and λ_1 denotes the relaxation parameter.

3. Heat Transfer Analysis

In this section, we are going to formulate heat transfer problem. Unlike typical studies, we derive governing equation of heat transfer using Cattaneo-Christov heat flux model [16]. According to this model the heat flux and temperature gradient are related through following expression

$$\mathbf{q} + \lambda_2 \left[\frac{\partial \mathbf{q}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{q} - \mathbf{q} \cdot \nabla \mathbf{V} + (\nabla \cdot \mathbf{V}) \mathbf{q} \right] = -k \nabla T, \quad (4)$$

Where \mathbf{q} represents the heat flux, λ_2 denotes the relaxation time of the heat flux, \mathbf{V} is the velocity vector, T is the Maxwell fluid temperature, k is the thermal conductivity. Eq. (4) reduces to well-known Fourier law for $\lambda_2 = 0$. For incompressible fluid $\nabla \cdot \mathbf{V} = 0$ and therefore Eq.(4) becomes

$$\mathbf{q} + \lambda_2 \left[\frac{\partial \mathbf{q}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{q} - \mathbf{q} \cdot \nabla \mathbf{V} \right] = -k \nabla T. \quad (5)$$

The energy equation for incompressible fluid after neglecting viscous dissipation effects is

$$\rho c_p \left[\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T \right] = -\nabla \cdot \mathbf{q}, \quad (6)$$

where c_p is the specific heat. Elimination of \mathbf{q} from Eq. (5) and Eq. (6) yields to the following single equation for the temperature field [16]

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial \bar{x}} + v \frac{\partial T}{\partial \bar{y}} + \lambda_2 \left[\begin{aligned} & \frac{\partial^2 T}{\partial t^2} + 2u \frac{\partial^2 T}{\partial \bar{x} \partial t} + 2v \frac{\partial^2 T}{\partial \bar{y} \partial t} + \frac{\partial v}{\partial t} \frac{\partial T}{\partial \bar{y}} + u \frac{\partial u}{\partial \bar{x}} \frac{\partial T}{\partial \bar{x}} + v \frac{\partial v}{\partial \bar{y}} \frac{\partial T}{\partial \bar{y}} \\ & + u \frac{\partial v}{\partial \bar{x}} \frac{\partial T}{\partial \bar{y}} + v \frac{\partial u}{\partial \bar{y}} \frac{\partial T}{\partial \bar{x}} + 2uv \frac{\partial^2 T}{\partial \bar{x} \partial \bar{y}} + u^2 \frac{\partial^2 T}{\partial \bar{x}^2} + v^2 \frac{\partial^2 T}{\partial \bar{y}^2} \end{aligned} \right] = \alpha \frac{\partial^2 T}{\partial \bar{y}^2}, \quad (7)$$

where $\alpha = k / \rho c_p$ is thermal diffusivity. Eq. (7) is subjected to the convective boundary conditions given by

$$-k \frac{\partial T}{\partial \bar{y}} = h(T_f - T), \quad \text{at} \quad \bar{y} = 0, \quad t > 0, \quad T \rightarrow T_\infty \quad \text{as} \quad \bar{y} \rightarrow \infty, \quad (8)$$

where h denotes the heat transfer coefficient. Moreover, T_f and T_∞ are the convective fluid temperatures below the surface and ambient fluid temperature, respectively.

3.1 Dimensionless formulation

Before going to the solution of problem, it is better to reduce the number of independent variables in Eqs. (2) and (7). We introduce following dimensionless quantities [44]

$$y = \sqrt{\frac{b}{\nu}} \bar{y}, \quad \tau = t\omega, \quad u = b\bar{x}f_y(y, \tau), \quad v = -\sqrt{\nu b}f(y, \tau), \quad (9)$$

$$\theta(y, \tau) = \frac{T - T_\infty}{T_f - T_\infty}. \quad (10)$$

Using Eqs. (9) and (10), the continuity equation is identically satisfied and Eqs.(2) and (7) are transformed into following forms

$$f_{yyy} - \beta \left[\begin{array}{l} S^2 f_{y\tau\tau} + 2S(f_y f_{y\tau} - ff_{yy\tau}) \\ -2ff_y f_{yy} + f^2 f_{yyy} \end{array} \right] - S(1 + \beta M^2) f_{y\tau} - f_y^2 - M^2 f_y + (1 + M^2 \beta) ff_{yy} = 0, \quad (11)$$

$$\frac{1}{\text{Pr}} \theta_{yy} + f \theta_y - S \theta_\tau - \gamma (S^2 \theta_{\tau\tau} - 2Sf \theta_{y\tau} - Sf_\tau \theta_y + ff_y \theta_y + f^2 \theta_{yy}) = 0. \quad (12)$$

Similarly, the boundary conditions of problem under consideration become

$$f_y(0, \tau) = \sin \tau, \quad f(0, \tau) = 0, \quad \theta_y(0, \tau) = -\gamma_1 [1 - \theta(0, \tau)], \quad f_y(\infty, \tau) = 0, \quad \theta(\infty, \tau) = 0, \quad (13)$$

where $M = \sqrt{\sigma B_0^2 / \rho b}$ represents the Hartmann number, $\lambda = \lambda_1 b$ denotes Deborah number, $\gamma = \lambda_2 b$ is the dimensionless relaxation time of heat flux, $S = \omega / b$ is ratio of oscillation frequency to stretching rate, $\gamma_1 = (h/k)\sqrt{\nu/b}$ represent the Biot number and $\text{Pr} = \nu/\alpha$ is the Prandtl number. Eq. (12) represents the energy equation based on Fourier law for $\gamma = 0$. For steady flow, Eq. (11) reduces to

$$f_{yyy} - \beta [f^2 f_{yyy} - 2ff_y f_{yy}] - f_y^2 - M^2 f_y + (1 + M^2 \beta) ff_{yy} = 0. \quad (27)$$

Moreover, for $\beta = 0$ it reduces to corresponding equation for flow of hydromagnetic viscous fluid. The corresponding equation for flow of hydrodynamic viscous fluid can be recovered by taking $\beta = M = 0$ [44].

4. Homotopy analysis method

The dimensionless partial differential equations (11)-(12) are highly nonlinear in nature and therefore exact solution is difficult to obtain. Therefore, we implement homotopy analysis method to compute series solution of these partial differential equations subject to boundary conditions (13). This method was originally proposed by Liao [45] and then successfully applied by many researchers in various disciplines of science and engineering [46-49]. To proceed with the solution, we suggest following initial approximations

$$f_0(y, \tau) = \sin \tau (1 - \exp(-y)), \quad \theta_0(y) = \frac{\gamma_1 \exp(-y)}{1 + \gamma_1}. \quad (14)$$

The auxiliary linear operators are

$$\mathcal{L}_f(f) = \frac{\partial^3 f}{\partial y^3} - \frac{\partial f}{\partial y}, \quad \mathcal{L}_\theta(f) = \frac{\partial^2 f}{\partial y^2} - f, \quad (15)$$

satisfying

$$\mathcal{L}_f[A_1 + A_2 \exp(y) + A_3 \exp(-y)] = 0, \quad (16)$$

$$\mathcal{L}_\theta[A_4 \exp(y) + A_5 \exp(-y)] = 0, \quad (17)$$

where A_i ($i = 1, 2, \dots, 5$) represent constants. The zeroth-order deformation problems for give problem is $(1-p)\mathcal{L}_f[\hat{f}(y, \tau; p) - f_0(y, \tau) = ph_f N_f[\hat{f}(y, \tau; p)]$, (18)

$$(1-p)\mathcal{L}_\theta[\hat{\theta}(y, \tau; p) - \theta_0(y, \tau) = ph_\theta N_\theta[\hat{f}(y, \tau; p), \hat{\theta}(y, \tau; p)]], \quad (19)$$

$$\hat{f}(y, \tau; p)\Big|_{y=0} = 0, \quad \frac{\partial \hat{f}(y, \tau; p)}{\partial y}\Big|_{y=0} = \sin \tau, \quad \frac{\partial \hat{f}(y, \tau; p)}{\partial y}\Big|_{y=\infty} = 0. \quad (20)$$

$$\frac{\partial \hat{\theta}(0, \tau; p)}{\partial y} = -\gamma_1 (1 - \hat{\theta}(0, \tau; p)), \quad \hat{\theta}(\infty, \tau; p) = 0, \quad (21)$$

The nonlinear operators are

$$\begin{aligned} N_f[\hat{f}(y, \tau; p)] = & \frac{\partial^3 \hat{f}(y, \tau; p)}{\partial y^3} - S(1 + \beta M^2) \frac{\partial^2 \hat{f}(y, \tau; p)}{\partial y \partial \tau} - M^2 \frac{\partial \hat{f}(y, \tau; p)}{\partial y} \\ & - \left(\frac{\partial \hat{f}(y, \tau; p)}{\partial y} \right)^2 + (1 + \beta M^2) \hat{f}(y, \tau; p) \frac{\partial^2 \hat{f}(y, \tau; p)}{\partial y^2} \\ & - \beta \left\{ \begin{aligned} & S^2 \frac{\partial^3 \hat{f}(y, \tau; p)}{\partial y \partial \tau^2} + 2S \left(\frac{\partial \hat{f}(y, \tau; p)}{\partial y} \frac{\partial^2 \hat{f}(y, \tau; p)}{\partial \tau \partial y} - \hat{f}(y, \tau; p) \frac{\partial^3 \hat{f}(y, \tau; p)}{\partial \tau \partial y^2} \right) \\ & + \hat{f}^2(y, \tau; p) \frac{\partial^3 \hat{f}(y, \tau; p)}{\partial y^3} - 2\hat{f}(y, \tau; p) \frac{\partial \hat{f}(y, \tau; p)}{\partial y} \frac{\partial^2 \hat{f}(y, \tau; p)}{\partial y^2} \end{aligned} \right\}, \quad (22) \end{aligned}$$

$$\begin{aligned} N_\theta[\hat{\theta}(y, \tau; p), \hat{f}(y, \tau; p)] = & \frac{1}{Pr} \frac{\partial^2 \hat{\theta}(y, \tau; p)}{\partial y^2} + \hat{f}(y, \tau; p) \frac{\partial \hat{\theta}(y, \tau; p)}{\partial y} - S \frac{\partial \hat{\theta}(y, \tau; p)}{\partial \tau} \\ & - \gamma \left\{ \begin{aligned} & S^2 \frac{\partial^2 \hat{\theta}(y, \tau; p)}{\partial \tau^2} - 2S \hat{f}(y, \tau; p) \frac{\partial^2 \hat{\theta}(y, \tau; p)}{\partial y \partial \tau} - S \frac{\partial \hat{f}(y, \tau; p)}{\partial \tau} \frac{\partial \hat{\theta}(y, \tau; p)}{\partial y} \\ & + \hat{f}(y, \tau; p) \frac{\partial \hat{f}(y, \tau; p)}{\partial y} \frac{\partial \hat{\theta}(y, \tau; p)}{\partial y} + \hat{f}^2(y, \tau; p) \frac{\partial^2 \hat{\theta}(y, \tau; p)}{\partial y^2} \end{aligned} \right\}. \quad (23) \end{aligned}$$

The solution of zeroth-order deformation problems at $p = 0$ and $p = 1$ is

$$f(y, \tau; 0) = f_0(y, \tau), \quad f(y, \tau; 1) = f(y, \tau), \quad (24)$$

$$\hat{\theta}(y, \tau; 0) = \theta_0(y, \tau), \quad \hat{\theta}(y, \tau; 1) = \theta(y, \tau). \quad (25)$$

Using Taylor's series expansion, we have

$$\hat{f}(y, \tau; p) = f_0(y, \tau) + \sum_{m=1}^{\infty} f_m(y, \tau) p^m, \quad f_m(y, \tau) = \frac{1}{m!} \frac{\partial^m \hat{f}(y, \tau; p)}{\partial p^m}, \quad (26)$$

$$\hat{\theta}(y, \tau; p) = \theta_0(y, \tau) + \sum_{m=1}^{\infty} \theta_m(y, \tau) p^m, \quad \theta_m(y, \tau) = \frac{1}{m!} \frac{\partial^m \hat{\theta}(y, \tau; p)}{\partial p^m}, \quad (27)$$

The convergence of HAM depends upon h_f and h_θ . We assume that h_f and h_θ are selected so that series solution converges at $p = 1$. Therefore

$$f(y, \tau; p) = f_0(y, \tau) + \sum_{m=1}^{\infty} f_m(y, \tau), \quad (28)$$

$$\theta(y, \tau) = \theta_0(y, \tau) + \sum_{m=1}^{\infty} \theta_m(y, \tau), \quad (29)$$

where f_m and θ_m can be computed through the m th-order of deformation problem given by

$$\mathcal{L}_f [f_m(y, \tau) - \chi_m f_{m-1}(y, \tau)] = h_f R_m^f(y, \tau), \quad (30)$$

$$\mathcal{L}_\theta [\theta_m(y, \tau) - \chi_m \theta_{m-1}(y, \tau)] = h_\theta R_m^\theta(y, \tau), \quad (31)$$

$$f_m(0, \tau) = 0, \quad \frac{\partial f_m(0, \tau)}{\partial y} = 0, \quad \frac{\partial f_m(\infty, \tau)}{\partial y} = 0, \quad (32)$$

$$\frac{\partial \theta_m(0, \tau)}{\partial y} - \gamma_1 \theta_m(0, \tau) = \theta_m(\infty, \tau) = 0, \quad (33)$$

$$\begin{aligned} R_m^f(y, \tau) = & \frac{\partial^3 f_{m-1}}{\partial y^3} - S(1 + \beta M^2) \frac{\partial^2 f_{m-1}}{\partial y \partial \tau} - M^2 \frac{\partial f_{m-1}}{\partial y} - \beta S^2 \frac{\partial^3 f_{m-1}}{\partial y \partial \tau^2} + (1 + M^2 \beta) \sum_{k=0}^{m-1} f_{m-1-k} \frac{\partial^2 f_k}{\partial y^2} - \sum_{k=0}^{m-1} \left(\frac{\partial f_{m-1-k}}{\partial y} \frac{\partial f_k}{\partial y} \right) + \\ & - \beta \sum_{k=0}^{m-1} \left\{ \left(2S \frac{\partial f_{m-1-k}}{\partial y} \frac{\partial^2 f_k}{\partial y \partial \tau} - 2S f_{m-1-k} \frac{\partial^3 f_k}{\partial \tau \partial y^2} \right) - f_{m-1-k} \sum_{l=0}^k \left(2 \frac{\partial f_{k-l}}{\partial y} \frac{\partial^2 f_l}{\partial y^2} - f_{k-l} \frac{\partial^3 f_l}{\partial y^3} \right) \right\}, \end{aligned} \quad (34)$$

$$\begin{aligned} R_m^\theta(y, \tau) = & \frac{1}{Pr} \frac{\partial^2 \theta_{m-1}}{\partial y^2} - S \frac{\partial \theta_{m-1}}{\partial \tau} - \gamma S^2 \frac{\partial^2 \theta_{m-1}}{\partial \tau^2} + \sum_{k=0}^{m-1} f_{m-1-k} \frac{\partial \theta_k}{\partial y} + S \gamma \sum_{k=0}^{m-1} \left(2 f_{m-1-k} \frac{\partial^2 \theta_k}{\partial y \partial \tau} + \frac{\partial f_{m-1-k}}{\partial \tau} \frac{\partial \theta_k}{\partial y} \right) \\ & - \gamma \sum_{k=0}^{m-1} \left\{ f_{m-1-k} \sum_{l=0}^k \left(\frac{\partial f_{k-l}}{\partial y} \frac{\partial \theta_l}{\partial y} + f_{k-l} \frac{\partial^2 \theta_l}{\partial y^2} \right) \right\}, \end{aligned} \quad (35)$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \quad (36)$$

The general solution is of the form

$$f_m(y, \tau) = f_m^*(y, \tau) + A_1 + A_2 \exp(y) + A_3 \exp(-y), \quad (37)$$

$$\theta_m(y, \tau) = \theta_m^*(y, \tau) + A_4 \exp(y) + A_5 \exp(-y). \quad (38)$$

where $f_m^*(y, \tau)$ and $\theta_m^*(y, \tau)$ represents the particular solution. Using (20) and (21), the constants A_i ($i=1, 2, \dots, 5$) are eliminated as

$$A_2 = A_4 = 0, \quad A_3 = \frac{\partial f_m^*(0, \tau)}{\partial y}, \quad A_1 = -A_3 - f_m^*(0, \tau), \quad A_5 = \frac{1}{\gamma_1 + 1} \left[\frac{\partial \theta_m^*(0, \tau)}{\partial y} - \gamma_1 \theta_m^*(0, \tau) \right]. \quad (39)$$

5. Discussion

The auxiliary parameters involved in the analytic expressions are h_f and h_θ . The convergence region as well as rate of approximations can be estimated for solutions by these parameters. The plots of $f_{yy}(0, \tau)$ versus h_f and $\theta_y(0, \tau)$ versus h_θ have been shown in Fig. 2 at 6th order of approximations to estimate suitable range of these parameter for convergent solutions. It is observed that for chosen set of involved parameters the admissible ranges for h_f and h_θ are: $-2 \leq h_f \leq -0.3$, $-1 \leq h_\theta \leq 0$, respectively. The plots of residual error for f and for a particular set of involved parameters at 6th order of approximation is shown in Fig. 3(a) when $h_f = -0.6$. Clearly, the maximum error over the whole domain is less than 6×10^{-6} . Similarly, Fig. 3(b) testifies that residual error for θ is in acceptable range when $h_f = -0.6$ and $h_\theta = -0.8$.

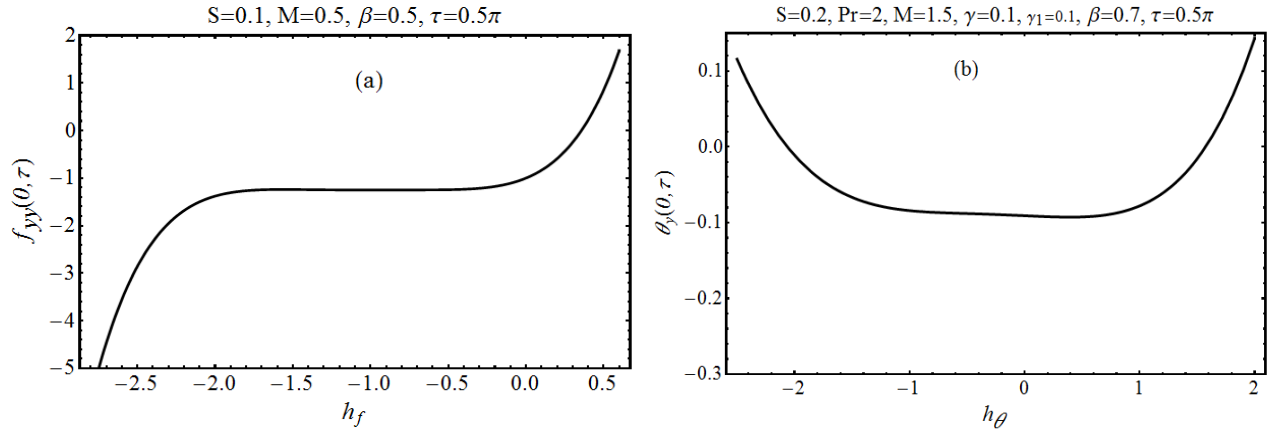


Fig. 2. h -curves for (a) velocity (b) temperature profiles.

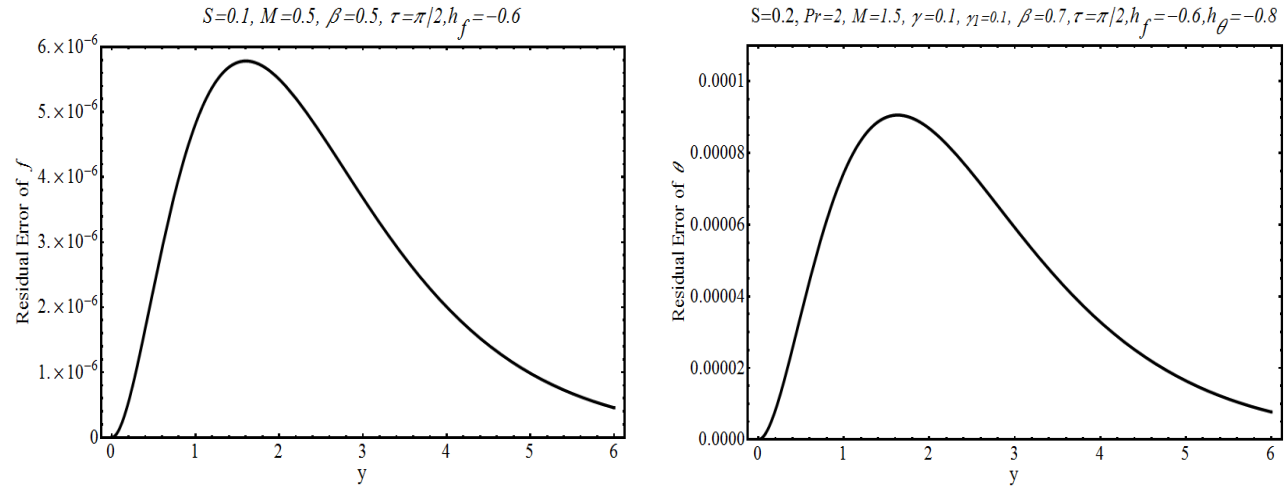


Fig. 3. Residual error for (a) velocity (b) temperature profiles.

Now we come to the discussion of graphical results concerning the velocity and temperature distribution for diverse values of various flow parameters like Hartmann number M , Deborah number β , relaxation time of the heat flux γ , Biot number γ_1 and Prandtl number Pr .

Fig.4(a) explains the effects of Deborah number β on the transverse velocity component f' by keeping $S = 0.2, M = 0.5$ and $\tau = \pi/2$. This figure shows that velocity decreases by increasing Deborah

number. For viscous fluid $\beta=0$ the momentum boundary layer is thicker as compared to non-Newtonian fluid. From physical and experimental point of view it is seen that at lower values of Deborah number, the fluid behaves much like liquid whereas the fluid shows viscoelastic solid like behavior at high Deborah number due to an increase in viscous properties and thus the fluid velocity starts to decelerate which is noted in Fig. 4(a). Moreover, the thickness of the boundary layer is suppressed for higher values of Deborah number.

The transverse distributions of the flow velocity f' for specific values Hartmann number M is shown in Fig. 4(b). Application of strong magnetic force tends to resist the velocity of fluid particles near the surface. This is in fact due to the fact that the presence of magnetic force produces the Lorentz force which resists the flow produced by oscillating sheet.

The velocity f' as function of time at a specific location $y=0.25$ is plotted for different values of Deborah number β and Hartmann number M in Fig. 5. A decrease in amplitude of flow velocity is noted with an increase in the Deborah number β (Fig. 5(a)). In fact for larger β , the viscous forces are dominant which restrict the motion of fluids particles and as a result amplitude is decreased. Fig.5(b) is sketched to examine the behavior of Hartmann number M on f' . Here, it is observed that the amplitude of the velocity f' decreases with increasing the values of Hartmann number M . Again this suppression in the amplitude is due to the resistive force produced due to application of magnetic field normal to the sheet.

The dependence of the fluid temperature on the Pr and γ is sketched in Fig. 6. Increasing the Prandtl number results in the thickening of thermal boundary layer. It can be justified because thermal diffusivity decreases for large values of Prandtl number which results in decrease of temperature and corresponding thermal boundary layer thickness. However, for non-zero values of γ the thermal boundary layer thickness decreases more rapidly with increasing Prandtl number.

In Fig. 7, we give the variation of temperature field θ for various values of γ and for two different values of Prandtl number. It is interesting to note that the temperature and thermal boundary layer thickness decreases with increasing γ . Further, it is observed that this decrease is faster for larger values of Prandtl number Pr . In Fig. 8 and 9, the effects of Deborah number β and Hartmann number M on temperature field are shown, respectively. Both the parameters effect the temperature field in a similar manner i.e. the temperature field increases by increasing either of β and M .

The variation of temperature profile for four different values of thermal Biot number $\gamma_1 = 1, 1.5, 2.5, 3.5$ are displayed in Fig. 10. Thermal Biot number is associated with heat transfer coefficient h , therefore its higher values represent the case of enhanced heat transfer from stretching sheet to the fluid stream. This enhancement in heat transfer is responsible for increase in the temperature of fluid. Fig.11 reflects the influence of ratio of oscillation frequency to stretching rate S on temperature profile θ . One can easily observe that temperature field θ is decreased by increasing S .

In Table 1, the obtained solution is validated against existing results of Zheng et al. [19] and Ali et al. [20]. An excellent agreement between both solutions is observed.

6. Concluding remarks

The heat transfer analysis in unsteady flow of Maxwell fluid by using Cattaneo-Christov heat flux model is presented when plate is stretched periodically. After computing the solution by homotopy analysis

method, a comprehensive analysis has been presented to highlight the effects of various flow parameters. The main findings of the analysis can be summarized as:

- By increasing Deborah number the fluid velocity is suppressed in the vicinity of the surface in given domain. The same is true with increasing Hartmann number.
- A oscillations in flow velocity at a specific location are suppressed for larger values of Deborah number and Hartmann number.
- It is noted that the heat transfer rate from sheet to the fluid become slow for larger values of Prandtl number and relaxation time of heat flux.
- The Cattaneo-Christov heat flux model predicts lower values of temperature inside the thermal boundary layer as compared heat flux model based on Fourier law.
- The rate of heat transfer enhanced by increasing Biot number γ_1 . Moreover, there is no heat transfer when $\gamma_1 = 0$.

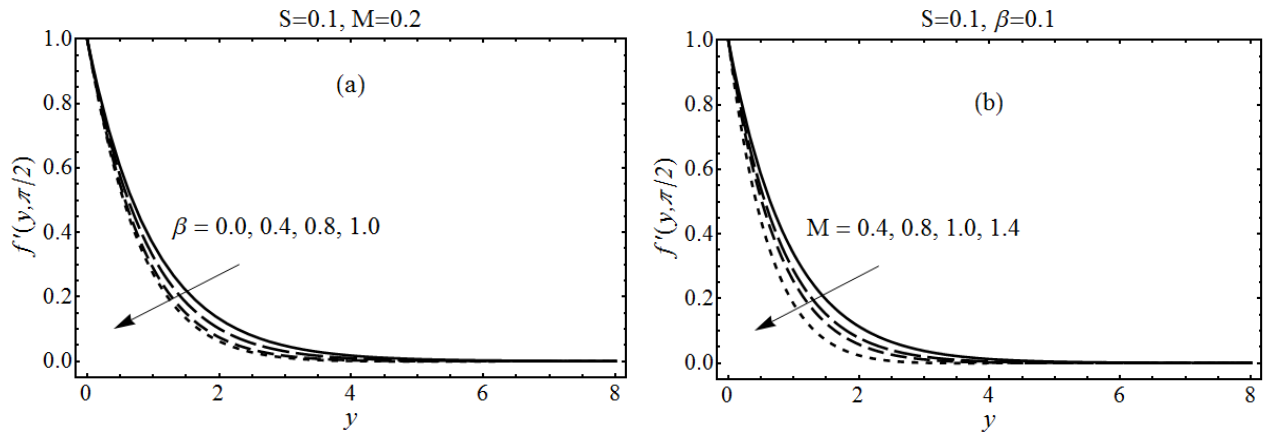


Fig. 4: The velocity profile for different values of (a) β (b) M

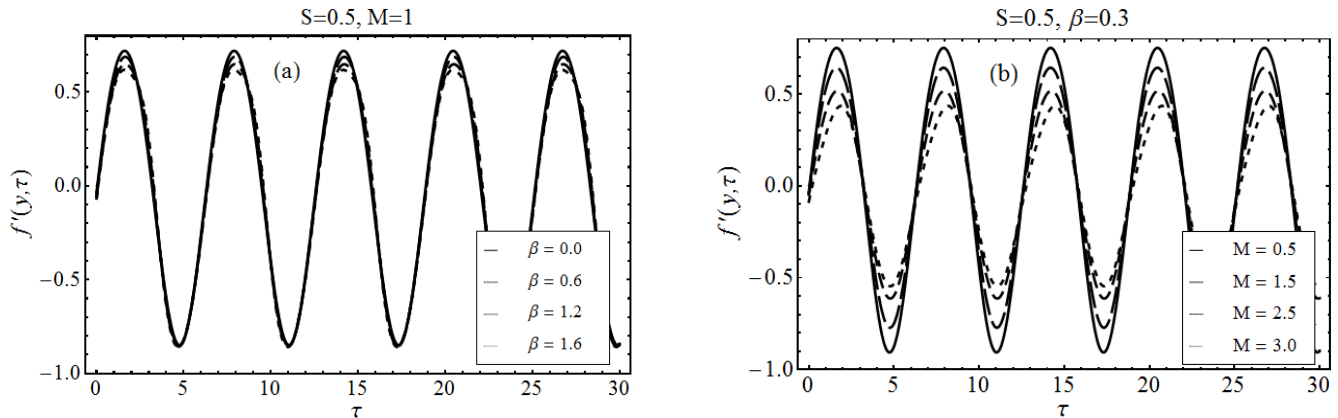


Fig. 5: Variation of velocity with time (a) effects of β and (b) effects of M

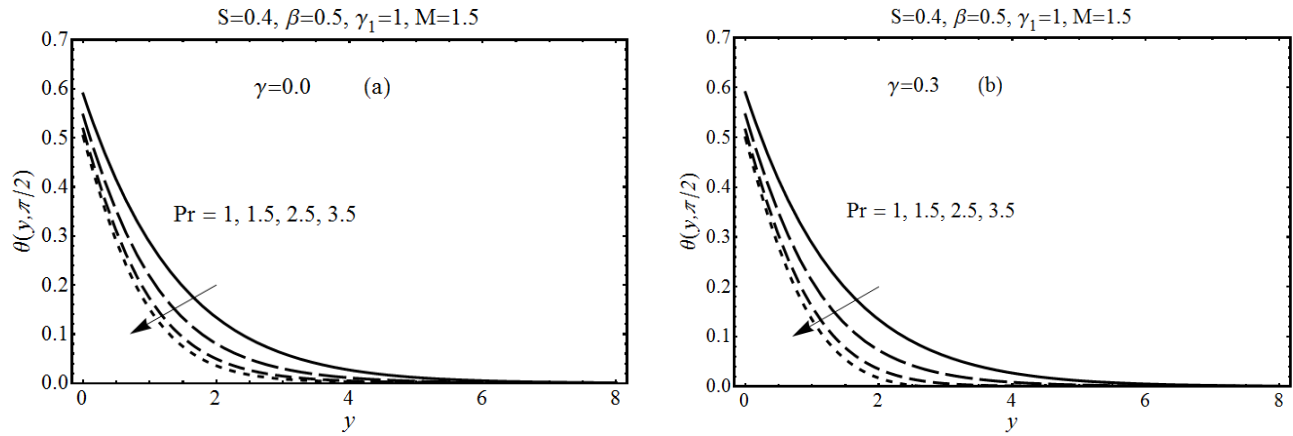


Fig. 6: Effects of Pr on θ

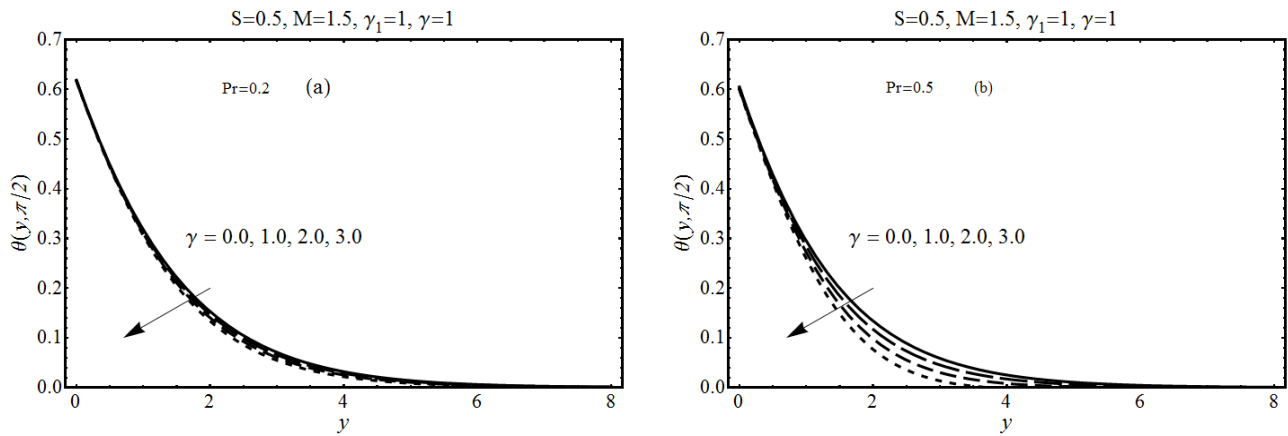


Fig. 7: Effects of γ on θ

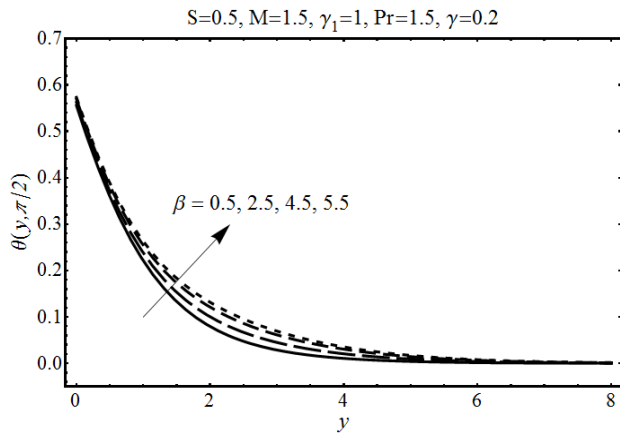


Fig. 8: Effects of β on θ

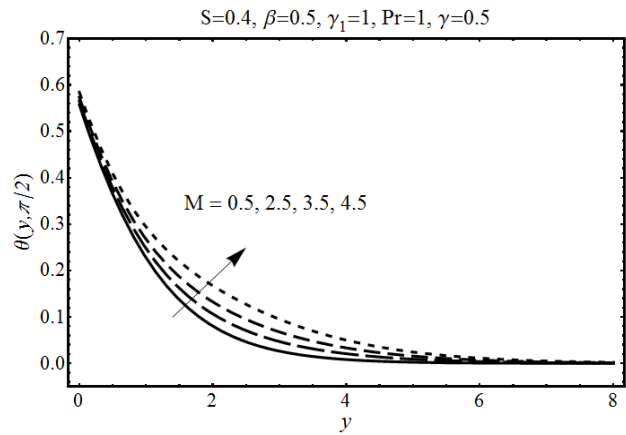


Fig. 9: Effects of M on θ

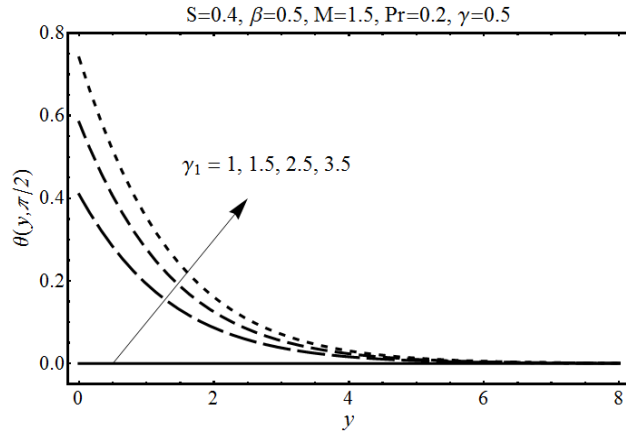


Fig. 10: Effects of γ_1 on θ

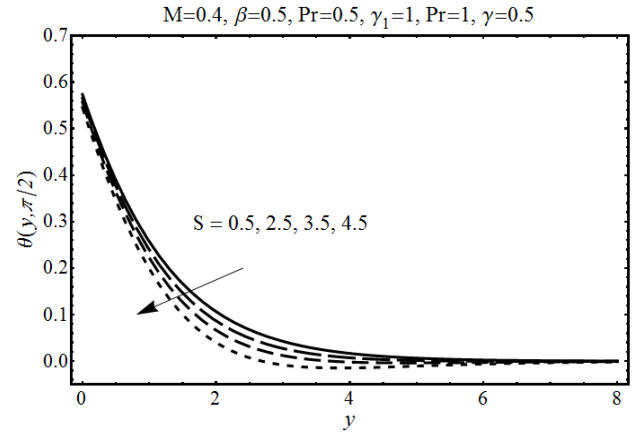


Fig. 11: Effects of S on θ

Table 1: Comparison of $f''(0, \tau)$ for various values of τ when $S = 1, M = 12$ in case of Newtonian fluid ($\beta = 0$).

τ	Zheng et al. [19]	Ali et al. [20]		Present Results
		HAM Results	Numerical Result	
1.5π	11.678656	11.678656	11.678656	11.678656
1.5π	11.678706	11.678706	11.678707	11.678706
1.5π	11.678656	11.678656	11.678656	11.678656

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