A FRACTIONAL MODEL FOR HEAT TRANSFER IN MONGOLIAN YURT

by

Hong-Yan LIU [a,b,c*]**, *Zhi-Min LI*** [d]**, and *Frank K. KO*** [c*]

[a] National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University, Suzhou, China
[b] School of Fashion Technology, Zhongyuan University of Technology, Zhengzhou, China
[c] Department of Materials Engineering, University of British Coloumbia, Vancouver, B. C., Canada
[d] Rieter Textile Instrument Co., Shanghai, China

Original scientific paper
https://doi.org/10.2298/TSCI150110081L

A yurt is a portable tent-like dwelling structure favored by Mongolian nomads for more than three millennia and it can be favorably used even at a harsh environment as low as –50 degrees. The paper concludes that the multi-layer structure of the felt cover is the key for weatherproofing. A fractional differential model with He's fractional derivative is established to find an optimal thickness of the fractal hierarchy of the felt cover. A better understanding of the yurt mechanism could help the further design of yurt-like space suits and other protective clothing for extreme cold region.

Key words: *fractional model, Mongolian yurt, variational iteration method*

Introduction

Yurts have been a distinctive feature of life in Central Asia for at least three thousand years [1-6]. The first written description of a yurt used as a dwelling was recorded by Herodotus (484~424 B. C.), a historical development of various yurts is systematically illustrated [1, 6]. The yurt is used by all nomadic Mongol and Turkish peoples and is now found in a huge territory spanning from Mongolia and Southern Siberia to Turkey. Few research works was reported on the thermal properties of yurts. Manfield [2] compared the thermal comfort of three different shelters used in cold climate for the design of expedition shelter. Contemporary yurts in some scenic spots near Hohhot city, China, however, have only geometrical similarity and many functional differences. It is interesting to note that the yurt has very few variations in form or detail within this area, yet it is notable for its mobility and climatic adaptability. The key to excellent thermal functionality lies in the felt making. Guo *et al.* [4] described a very detailed felt making process.

The yurt has both a flexible and thermally responsive insulating fabric, in the form of removable layers of felt that wrap around the wooden lattice and pole structure. During summer, the felt is rolled up to allow cross ventilation. In some parts of Asia, the felt is permanently replaced with woven reed mats in order to maintain privacy as well as thermal comfort. Each felt layer is between 20 and 30 mm thick and during winter months. A yurt may

* Corresponding author, e-mail: phdliuhongyan@yahoo.com

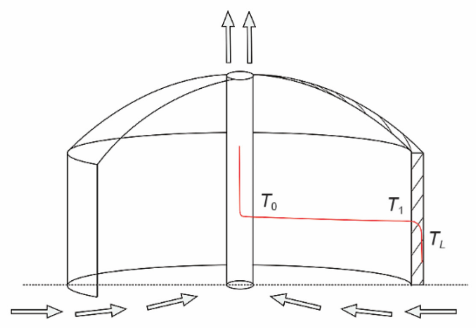

Figure 1. Mongolian yurt, and its temperature drop. The slope of the temperature at the inner wall tends to zero [6]

have as many as eight felt layers to minimize heat lost from the open hearth at the center of the floor plan [2]. The felt cover, which consists of layers of fabric and sheep wool, mimics the design of a cocoon. And other researchers investigated the thermal properties of tent clothing showing the importance of hierarchical structure [3, 4, 6] as that in a coccoon [7]. If the yurt were not weatherproofed, the people in it might have frozen to death in winter. The felt cover evolved over time. It is reported that the yurt felt cover possess hierarchical structure which is the major factor for the high protection function of yurt and clothing [5, 6]. A theoretical analysis was given to explain the fascinating phenomenon by a fractal hydrodynamic model for a discontinuous membrane composed of a hierarchical wool cascade. Mathematical model was proposed for the better understanding of heat transfer in the yurt [6].

As depicted in fig.1, the slope of the temperature change is almost zero near the inner wall of the yurt, while the temperature changes quickly near the outer wall which can only be achieved by fractional calculus. Fractional calculus can effectively describe phenomena arising in discontinuous media.

Mathematical models

The definition of fractional derivative is of mathematical importance and practical applications [8]. The following are some examples listed.

The Caputo fractional derivative is defined as [8]:

$$\mathrm{D}_x^\alpha[f(x)] = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^x (x-t)^{n-\alpha-1} \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\,\mathrm{d}t \tag{1}$$

Riemann-Liouville fractional derivative is defined [8]:

$$\mathrm{D}_x^\alpha[f(x)] = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int\limits_0^x (x-t)^{n-\alpha-1} f(t)\mathrm{d}t \tag{2}$$

Caputo derivatives are defined only for differentiable functions, while f can be a continuous (but not necessarily differentiable) function. The Riemann-Liouville definition can be used for any functions that are continuous but not differentiable anywhere, however, one requires that $\mathrm{D}_x^\alpha[f(x)] \neq 0$ when $f(x)$ is a constant.

To overcome the disadvantages, Jumarie [9] modified Riemann-Liouville fractional derivative into the form:

$$\mathrm{D}_x^\alpha[f(x)] = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int\limits_0^x (x-t)^{n-\alpha-1} [f(t) - f(0)]\mathrm{d}t \tag{3}$$

Alternative definitions were demonstrated in [8]. In this paper, we adopt He's definition [10]:

$$D_t^\alpha f = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1} [f(t_0) - f(s)] ds \tag{4}$$

In previous equations, f can be continuous and possibly not differentiable anywhere. The definition given in eq. (4) is derived from the variational iteration method [10], its physical understanding was elucidated [10], and practiced in many applications [11-17].

Fractal hierarchy and fractional model

The yurt have special hierarchical structures [3] which enable the yurt to have excellent protective properties. A yurt has a multi-layer structure and the fiber diameter reduces greatly from the outer layer to the inner layer. To illustrate the heat transfer property in a fractal hierarchy, we established a 1-D model using the local fractional calculus [8]. In this paper, we find that He's fractional model [10] is more suitable for our study. He's fractional derivative is specially effective for heat conduction in porous media, Liu *et al.* [11] applied He's fractional derivative for heat conduction in a fractal medium arising in silkworm cocoon hierarchy, and gave a mathematical explanation of cocoon thermal protection and applied to optimally design insulation clothing with cocoon-like porous structure [12]. Fan *et al.* [13] studied bio-mimic design of multi-scale fabrics according to the structure of wools, and found that such structure has extreme efficient heat transfer property and high air permeability [14]. Wang *et al.* [15, 16] and Sayevand *et al.* [17] gave some effective analytical methods for He's fractional calculus.

Fourier's law of thermal conduction inside the yurt and through a fractal hierarchy of its wall can be expressed using He's fractional derivative, respectively [10]:

$$\frac{d}{dx}\left(k_0 \frac{dT}{dx}\right) = 0, \qquad 0 < x < x_{L_1} \tag{5}$$

$$\frac{d^\alpha}{dx^\alpha}\left(k_1 \frac{d^\alpha T}{dx^\alpha}\right) = 0, \qquad x_{L_1} \le x \le L \tag{6}$$

with following boundary conditions:

$$T(0) = T_0 \tag{7}$$

$$T(L) = T_L \tag{8}$$

$$\left.\left(k_0 \frac{dT}{dx}\right)\right|_{x=L_1} = \left.\left(k_1 \frac{d^\alpha T}{dx^\alpha}\right)\right|_{x=L_1} \tag{9}$$

where $x = x_{L_1}$ is the inner wall, $x = L$ – the outer wall of the yurt, k_0 – the thermal conductivity of heat flux in air, k_1 – the thermal conductivity of heat flux in the fractal hierarchy, d^α/dx^α – the He's fractional derivative [10] instead of the local fractional partner [6, 8], and α – the fractal dimensions of fractal hierarchy.

The general solution to eq. (5) is:

$$T = T_0 + ax \tag{10}$$

where a is a constant.

The boundary condition, eq. (9), becomes:

$$\left.\left(k_1 \frac{\mathrm{d}^\alpha T}{\mathrm{d}x^\alpha}\right)\right|_{x=L_1} = k_0 a \tag{11}$$

To solve the fractional differential equation, eq. (6), with boundary conditions, eqs. (8) and (11), we adopt the following fractional complex transform [18-20]:

$$s = \frac{x^\alpha}{\Gamma(1+\alpha)} \tag{12}$$

Equations (6) and (11) become, respectively:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(k_1 \frac{\mathrm{d}T}{\mathrm{d}s}\right) = 0 \tag{13}$$

and

$$\left.\left(k_1 \frac{\mathrm{d}T}{\mathrm{d}s}\right)\right|_{s=L_1^\alpha/\Gamma(1+\alpha)} = k_0 a \tag{14}$$

The general solution of eq. (13) is:

$$T = b + cs \tag{15}$$

or equivalently:

$$T = b + \frac{c}{\Gamma(1+\alpha)} x^\alpha \tag{16}$$

where b and c are constants to be further determined.

Incorporating the boundary conditions with eqs. (8), (11), and (14), we have that:

$$T_L = b + \frac{c}{\Gamma(1+\alpha)} L^\alpha \tag{17}$$

$$k_1 c = k_0 a \tag{18}$$

$$T_0 + \frac{a}{\Gamma(1+\alpha)} L_1^\alpha = b + \frac{c}{\Gamma(1+\alpha)} L_1^\alpha \tag{19}$$

Solving eqs. (17)-(19) simultaneously, we can identify a, b, and c:

$$a = \frac{T_L - T_0}{\dfrac{k_0 (L^\alpha - L_1^\alpha)}{k_1 \Gamma(1+\alpha)} + \dfrac{L_1^\alpha}{\Gamma(1+\alpha)}} \tag{20}$$

$$b = T_L - \frac{(T_L - T_0) L^\alpha}{L^\alpha - L_1^\alpha + \dfrac{k_1}{k_0} L_1^\alpha} \tag{21}$$

$$c=\frac{T_L-T_0}{\dfrac{L^\alpha-L_1^\alpha}{\Gamma(1+\alpha)}+\dfrac{k_1}{k_0}\dfrac{L_1^\alpha}{\Gamma(1+\alpha)}} \tag{22}$$

We can, therefore, determine the temperature and its slope at the inner wall:

$$T_1=T(L_1)=T_0+\frac{aL_1^\alpha}{\Gamma(1+\alpha)}=T_0+\frac{(T_L-T_0)\dfrac{L_1^\alpha}{\Gamma(1+\alpha)}}{\dfrac{k_0(L^\alpha-L_1^\alpha)}{k_1\Gamma(1+\alpha)}+\dfrac{L_1^\alpha}{\Gamma(1+\alpha)}}=T_0+\frac{T_L-T_0}{\dfrac{k_0L^\alpha\left[1-\left(\dfrac{L_1}{L}\right)^\alpha\right]}{k_1L_1^\alpha}+1} \tag{23}$$

and

$$T_1^{(\alpha)}=\frac{\mathrm{d}^\alpha T}{\mathrm{d}x^\alpha}(x=L_1)=\frac{T_L-T_0}{(L^\alpha-L_1^\alpha)+\dfrac{k_1L_1^\alpha}{k_0}}=\frac{T_L-T_0}{L^\alpha+\left(\dfrac{k_1}{k_0}-1\right)L_1^\alpha} \tag{24}$$

It is obvious that T_1 should be closed to the body temperature and its slope, $T_1^{(\alpha)}$, should tend to zero (see fig. 1), which requires that:

$$T_1^{(\alpha)}=\frac{T_L-T_0}{L^\alpha+\left(\dfrac{k_1}{k_0}-1\right)L_1^\alpha}=0 \tag{25}$$

$$\frac{\mathrm{d}^{2\alpha}T_1}{\mathrm{d}L_1{}^{2\alpha}}=0 \tag{26}$$

$$T_{L_1}\approx T_{\text{body}} \tag{27}$$

From eqs. (25)-(27), L_1, k_1, and α can be optimally determined.

In most practical applications, $T_1^{(\alpha)}\neq 0$ or $\mathrm{d}^{2\alpha}T_1/\mathrm{d}L_1{}^{2\alpha}\neq 0$, L_1, k_1, and α should be such chosen that $\mathrm{d}^{2\alpha}T_1/\mathrm{d}L_1{}^{2\alpha}\neq 0$ or $T_1^{(\alpha)}\neq 0$ is small enough and tends to zero, the smaller, the better.

Conclusion

The paper highlights that the human-yurt system is a hierarchical structure. A local fractional model is established to illustrate its mechanism and to optimally determine the thickness of the felt cover. The fractal derivative is especially suitable for the hierarchical structure. Non-lienar fractal differential equation can be converted to ordinary differential equation which is solved accurately by variational iteration method. A better understanding of the yurt mechanism could help the further design of yurt-like space suits and other protective clothing for extreme cold region.

Acknowledgment

The work is supported by Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD), National Natural Science Foundation of China under

Grant No. 11372205 and No. 51576215 Jiangsu Planned Projects for Postdoctoral Research Funds under Grant No. 1401076B, China Postdoctoral Science Foundation under Grant Nos. 2015M571806 and 2016T90495, China National Textile and Apparel Council Project under Grant No. 2015011, Key Scientific Research Projects of Henan Province under Grant No. 16A540001, and program of China Scholarship Council and the support provided by the NSERC Discovery Grant is also appreciated.

References

[1] Humphrey, C., Inside a Mongolian Tent, *New Society, 30* (1974), 630, pp. 273-275
[2] Manfield, P., A Comparative Study of Temporary Shelters Used in Cold Climates, M. Ph. thesis, Cambridge University, Cambridge, UK, 2000
[3] Nielsen, R., *et al.*, Thermal Function of a Clothing Ensemble during Work: Dependency on Inner Clothing Layer Fit, *Ergonomics, 32* (1989), 12, pp. 1581-1594
[4] Guo, X. F., *et al.*, Study on Thermal Properties of Tibetan Robe Ensemble in Different Wearing Ways, *Advanced Materials Research, 332-334* (2011), Sept., pp. 367-370
[5] Li, J., *et al.*, Temperature Rating Prediction of Tibetan Robe Ensemble Based on Different Wearing Ways, *Applied Ergonomics, 43* (2012), 5, pp. 909-915
[6] Liu, H. Y., He, J.-H., From Leibniz's Notation for Derivative to the Fractal Derivative, Fractional Derivative and Application in Mongolian Yurt, in: *Fractional Dynamics* (Ed. C. Cattani, H. M. Srivastava, X.-J., Yang), De Gruyter, Berlin, 2016, pp. 219-228
[7] Chen, R., *et al.*, Silk Cocoon: "Emperor's New Clothes" for Pupa: Fractal Nano-Hydrodynamical Approach, *Journal of Nano Research, 22* (2013), May, pp. 65-70
[8] Yang, X. J., *Advanced Local Fractional Calculus and Its Applications,* World Science, New York, USA, 2012
[9] Jumarie, G., Fractional Partial Differential Equations and Modified Riemann-Liouville Derivative New Methods for Solution, *Journal of Applied Mathematics and Computing, 24* (2007), 1-2, pp. 31-48
[10] He, J.-H., A Tutorial Review on Fractal Spacetime and Fractional Calculus, *International Journal of Theoretical Physics, 53* (2014), 11, pp. 3698-3718
[11] Liu, F. J., *et al.*, He's Fractional Derivative for Heat Conduction in a Fractal Medium Arising in Silkworm Cocoon Hierarchy, *Thermal Science, 19* (2015), 4, pp. 1155-1159
[12] Liu, F. J., *et al*., A Fractional Model for Insulation Clothings with Cocoon-Like Porous Structure, *Thermal Science, 20* (2016), 3, pp. 779-784
[13] Fan, J., He, J.-H., Biomimic Design of Multi-Scale Fabric with Efficient Heat Transfer Property, *Thermal Science, 16* (2012), 5, pp. 1349-1352
[14] Fan, J., He, J.-H., Fractal Derivative Model for Air Permeability in Hierarchic Porous Media, *Abstract and Applied Analysis, 2012* (2012), ID 354701
[15] Wang, K. L., Liu, S. Y., A New Solution Procedure for Nonlinear Fractional Porous Media Equation Based on a New Fractional Derivative, *Nonlinear Science Letters A, 7* (2016), 4, pp. 135-140
[16] Wang, K. L., Liu, S. Y., He's Fractional Derivative for Nonlinear Fractional Heat Transfer Equation, *Thermal Science, 20* (2016), 3, pp. 793-796
[17] Sayevand, K., Pichaghchi, K., Analysis of Nonlinear Fractional KdV Equation Based on He's Fractional Derivative, *Nonlinear Science Letters A, 7* (2016), 3, pp. 77-85
[18] Li, Z. B., He, J.-H., Fractional Complex Transform for Fractional Differential Equations, *Mathematical & Computational Applications, 15* (2010), 5, pp. 970-973
[19] He, J.-H., Li, Z. B., Converting Fractional Differential Equations into Partial Differential Equations, *Thermal Science,16* (2012), 2, pp. 331-334
[20] He, J.-H., *et al*., Geometrical Explanation of the Fractional Complex Transform and Derivative Chain Rule for Fractional Calculus, *Physics Letters A, 376* (2012), 4, pp. 257-259

Paper submitted: January 10, 2015
Paper revised: May 5, 2016
Paper accepted: July 12, 2016