SOME NEW APPLICATIONS FOR HEAT AND FLUID FLOWS VIA FRACTIONAL DERIVATIVES WITHOUT SINGULAR KERNEL

by

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This paper addresses the mathematical models for the heat-conduction equations and the Navier-Stokes equations via fractional derivatives without singular kernel.

Key words: *heat-conduction equation, Navier-Stokes equation, fractional derivatives without singular kernel*

Introduction

Fractional derivatives of variable order [1-4] has used to set up the mathematical models for engineering practice, especially in the fields of the heat [5, 6] and fluid flows [7, 8].

Recently, Caputo and Fabrizio [9] reported the fractional derivative operator without singular kernel, which was given [10-13]:

$$\mathrm{D}_x^{(\beta)}\Xi(x) = \frac{(2-\beta)\aleph(\beta)}{2(1-\beta)}\int_0^x \exp\left[-\frac{\beta}{1-\beta}(x-\lambda)\right]\Xi^{(1)}(\lambda)\mathrm{d}\lambda \tag{1}$$

where $\aleph(\beta)$ is a normalization constant depending on β $(0<\beta<1)$, such that $\aleph(0)=\aleph(1)=1$.

More recently, Yang *et al.* [12, 13] reported a new fractional derivative without singular kernel:

$$\mathrm{D}_{a^+}^{(\beta)}\Omega(x) = \frac{\aleph(\beta)}{1-\beta}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \exp\left[-\frac{\beta}{1-\beta}(x-\lambda)\right]\Omega(\lambda)\mathrm{d}\lambda \tag{2}$$

where $a \le x$, β $(0<\beta<1)$ is a real number, and $\aleph(\beta)$ is a normalization function depending on β such that $\aleph(0)=\aleph(1)=1$.

In this article, our aim is to set up the heat-conduction equation and the Navier-Stokes equation via fractional derivatives without singular kernel.

Mathematical tools

The fractional gradient operator via the Caputo and Fabrizio [9] fractional derivative without singular kernel is given by:

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$$ {}_{CF}\nabla^{(\beta)}\wp(x) =: \frac{\beta}{(1-\beta)\sqrt{\pi^{\beta}}} \int_{\Omega} \nabla\,\wp(x)\,\exp\left\{-\left[\frac{\beta}{1-\beta}(x-y)\right]^2\right\}\mathrm{d}y \tag{3a} $$

where $x, y \in \Omega$.

The fractional tensor via the Caputo and Fabrizio [9] fractional derivative without singular kernel is given by:

$$ {}_{CF}\nabla^{(\beta)}A(x) =: \frac{\beta}{(1-\beta)\sqrt{\pi^{\beta}}} \int_{\Omega} \nabla A(x)\,\exp\left\{-\left[\frac{\beta}{1-\beta}(x-y)\right]^2\right\}\mathrm{d}y \tag{3b} $$

where $x, y \in \Omega$.

The fractional Laplacian operator via the Caputo and Fabrizio [9] fractional derivative without singular kernel is given by:

$$ {}_{CF}\nabla^{(2\beta)}\wp(x) =: \frac{\beta}{(1-\beta)\sqrt{\pi^{\beta}}} \int_{\Omega} \nabla^2\wp(x)\,\exp\left\{-\left[\frac{\beta}{1-\beta}(x-y)\right]^2\right\}\mathrm{d}y \tag{3c} $$

where $x, y \in \Omega$.

The fractional curl via the Caputo and Fabrizio [9] fractional derivative without singular kernel is given by:

$$ {}_{CF}\nabla^{(\beta)}\times A(x) =: \frac{\beta}{(1-\beta)\sqrt{\pi^{\beta}}} \int_{\Omega} \nabla\times A(x)\,\exp\left\{-\left[\frac{\beta}{1-\beta}(x-y)\right]^2\right\}\mathrm{d}y \tag{3d} $$

where $x, y \in \Omega$.

The fractional gradient operator via the new fractional derivative without singular kernel is defined:

$$ \nabla^{(\beta)}\wp(x) =: \nabla\left(\frac{\beta\Im(\beta,\mathrm{n})}{(1-\beta)\sqrt{\pi^{\beta}}} \int_{\Omega} \wp(x)\,\exp\left\{-\left[\frac{\beta}{1-\beta}(x-y)\right]^2\right\}\mathrm{d}y\right) \tag{4a} $$

where $x, y \in \Omega \in \mathbb{R}^n$.

We have:

$$ \begin{aligned} &\lim_{\beta\to 1}\left(\int_{\Omega} \wp(x)\,\exp\left\{-\left[\frac{\beta}{1-\beta}(x-y)\right]^2\right\}\mathrm{d}y/[\pi(1-\beta)/\beta]^{\frac{n}{2}}\right) \\ &= \lim_{\beta\to 1}\int_{\Omega}\wp(x)\delta(x-y)\mathrm{d}y \\ &= \wp(x) \end{aligned} \tag{4b} $$

such that:

$$ \lim_{\beta\to 1}\frac{\beta^{\frac{n+2}{2}}\Im(\beta,n)}{(1-\beta)^{\frac{n+2}{2}}\pi^{\frac{n+\beta}{2}}} = 1 \tag{4c} $$

where $x, y \in \Omega \in \mathbb{R}^n$ and $\Im(\beta, n)$ is a constant.

From eq. (4c), we have the following property:

$$\lim_{\beta\to 1} \nabla^{(\beta)}\wp(x) = \nabla\,\wp(x) \tag{4d}$$

The fractional tensor via the new fractional derivative without singular kernel is defined:

$$\nabla^{(\beta)}A(x) =: \nabla\left(\frac{\beta\Im(\beta,n)}{(1-\beta)\sqrt{\pi^{\beta}}}\int_{\Omega}\wp(x)\exp\left\{-\left[\frac{\beta}{1-\beta}(x-y)\right]^2\right\}\mathrm{d}y\right) \tag{4e}$$

where $x, y \in \Omega \in \mathbb{R}^n$ and $\Im(\beta, n)$ is a constant.

The fractional Laplacian operator via the new fractional derivative without singular kernel is defined:

$$\nabla^{(2\beta)}\wp(x) = \nabla^2\left(\frac{\beta\Im(\beta,n)}{(1-\beta)\sqrt{\pi^{\beta}}}\int_{\Omega}\wp(x)\exp\left\{-\left[\frac{\beta}{1-\beta}(x-y)\right]^2\right\}\mathrm{d}y\right) \tag{4f}$$

where $x, y \in \Omega \in \mathbb{R}^n$ and $\Im(\beta, n)$ is a constant.

The fractional curl via the new fractional derivative without singular kernel is given by:

$$\nabla^{(\beta)}\times A(x) =: \nabla\times\left(\frac{\beta\Im(\beta,n)}{(1-\beta)\sqrt{\pi^{\beta}}}\int_{\Omega}\wp(x)\exp\left[-\left[\frac{\beta}{1-\beta}(x-y)\right]^2\right]\mathrm{d}y\right) \tag{4g}$$

where $x, y \in \Omega \in \mathbb{R}^n$ and $\Im(\beta, n)$ is a constant.

In eq. (4c), for $n = 1$, $n = 2$, and $n = 3$, we have:

$$\Im(\beta,1) = \frac{(1-\beta)^{\frac{3}{2}}\pi^{\frac{1+\beta}{2}}}{\beta^{\frac{3}{2}}}, \qquad \Im(\beta,2) = \frac{(1-\beta)^2\pi^{\frac{2+\beta}{2}}}{\beta^2}, \qquad \Im(\beta,3) = \frac{(1-\beta)^{\frac{5}{2}}\pi^{\frac{3+\beta}{2}}}{\beta^{\frac{5}{2}}} \tag{4h}$$

According to the expressions (4f) and (4g), we directly have the properties:

$$\nabla^{(2\beta)}\wp(x) = \nabla\nabla^{(\beta)}\wp(x), \qquad \lim_{\beta\to 1}\nabla^{(\beta)}A(x) = \nabla A(x) \tag{4i,j}$$

In order to discuss the problems, we replace the operators $_{CF}\nabla$ and ∇ by $^{*}\nabla$ in this article.

The heat-conduction problem via fractional derivatives without singular kernel

Following the idea [14-16], the Fourier law of the heat conduction via fractional derivatives without singular kernel is expressed by:

$$\vec{\mathrm{q}}(x, y, z, \tau) = -\kappa\,{}^{*}\nabla^{(\beta)}T(x, y, z, \tau) \tag{5a}$$

where $\vec{\mathrm{q}}(x, y, z, \tau)$ is the heat flow, κ – the thermal conductivity of the material, and ${}^{*}\nabla^{(\beta)}$ – the fractional gradient operator via the fractional derivatives without singular kernel. The Fourier law of the heat conduction via fractional derivatives without singular kernel in 1-D space was discussed in [12].

The fractional heat-conduction equation with heat generation via fractional derivatives without singular kernel is written in the form:

$$\kappa\,{}^{*}\nabla^{(2\beta)}T(x, y, z, \tau) - \rho c\,\frac{\partial T(x, y, z, \tau)}{\partial t} + g(x, y, z, \tau) = 0 \tag{5b}$$

where $g(x, y, z, \tau)$ is the volume heat generation, ρ and c are the density and the specific heat of the material, respectively.

The fractional heat-conduction equations within fractional derivatives without singular kernel in the 2-D case read:

$$\kappa\,{}^{*}\nabla^{(2\beta)}T(x, y, \tau) - \rho c\,\frac{\partial T(x, y, \tau)}{\partial t} + g(x, y, \tau) = 0 \tag{6}$$

where $g(x, y, \tau)$ is the area heat generation, ${}^{*}\nabla^{(2\beta)}$ – the corresponding fractional gradient operators via the fractional derivatives without singular kernel, and κ denotes the thermal conductivity of the material.

The Navier-Stokes equations via fractional derivatives without singular kernel

We now structure the fractional velocity gradient tensor in the form:

$${}^{*}\nabla^{(\beta)}\upsilon = \frac{1}{2}(\Theta + \Theta^{T}) + \frac{1}{2}(\Theta + \Theta^{T}) = \mathbf{\Lambda} + \frac{1}{2}(\Theta + \Theta^{T}) \tag{7a}$$

which leads to the fractional strain rate tensor can be written as:

$$\mathbf{\Lambda} = \frac{1}{2}\left[{}^{*}\nabla^{(\beta)}\upsilon + \upsilon\,{}^{*}\nabla^{(\beta)}\right], \tag{7b}$$

where υ is the fluid velocity, $\Theta = {}^{*}\nabla^{(\beta)}\upsilon$ and $\Theta^{T} = \upsilon\,{}^{*}\nabla^{(\beta)}$.

Following eqs. (7a) and (7b), we can structure the linear relation of the type of fractional Cauchy stress:

$$\mathbf{J} = -p\vec{\mathrm{I}} + 2\mu\mathbf{\Lambda} + \lambda[{}^{*}\nabla^{(\beta)}\upsilon]\vec{\mathrm{I}} \tag{8}$$

where p is the thermodynamic pressure, $\mathbf{\Lambda}$ – the strain rate tensor, $\vec{\mathrm{I}}$ – unit vector in the field, and λ and μ the bulk and shear moduli of viscosity, respectively.

Following the idea in [16], we write the continuity equation of the fractional flow in the form:

$$\frac{\partial \rho}{\partial t} + \upsilon[{}^{*}\nabla^{(\beta)}\rho] = 0 \tag{9}$$

where ρ represents the fluid density.

Similarly, we give the Cauchy's equation of motion of the fractional flows:

$$\rho\,\frac{\partial \upsilon}{\partial t} = {}^{*}\nabla^{(\beta)}\mathbf{J} + \rho b - \rho[\upsilon\,{}^{*}\nabla^{(\beta)}]\upsilon \tag{10}$$

where b is the specific body force and $[\upsilon^{*}\nabla^{(\beta)}]\upsilon$ – the convection term of the fractional flow.

From eqs. (9) and (10) the systems of the fractional Navier-Stoke equations are given by:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \upsilon[^{*}\nabla^{(\beta)}\rho] = 0, \\ \rho\dfrac{\partial \upsilon}{\partial t} = {}^{*}\nabla^{(\beta)}\{-p\bar{\mathrm{I}} + 2\mu\mathbf{\Lambda} + \lambda[^{*}\nabla^{(\beta)}\upsilon]\bar{\mathrm{I}}\} + \rho b - \rho[\upsilon^{*}\nabla^{(\beta)}]\upsilon, \\ \upsilon = \upsilon_0 \end{cases} \tag{11a}$$

which leads to:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \upsilon[^{*}\nabla^{(\beta)}\rho] = 0, \\ \rho\dfrac{\partial \upsilon}{\partial t} = -{}^{*}\nabla^{(\beta)}p\bar{\mathrm{I}} + {}^{*}\nabla^{(\beta)}2\mu\mathbf{\Lambda} + {}^{*}\nabla^{(\beta)}\lambda[^{*}\nabla^{(\beta)}\upsilon]\bar{\mathrm{I}} + \rho b - \rho[\upsilon^{*}\nabla^{(\beta)}]\upsilon, \\ \upsilon = \upsilon_0 \end{cases} \tag{11b}$$

The constitutive equation of the incompressible Navier-Stokes fluid can be written in the form:

$$\mathbf{J} = -p\bar{\mathrm{I}} + 2\mu\mathbf{\Lambda}, \quad {}^{*}\nabla^{(\beta)}\upsilon = 0 \tag{12a,b}$$

since we present:

$$\frac{\partial \rho}{\partial t} + \upsilon[^{*}\nabla^{(\beta)}\rho] = \frac{\partial \rho}{\partial t} + {}^{*}\nabla^{(\beta)}(\upsilon\rho) \tag{12c}$$

In view of eqs. (12a) and (12b), eq. (11b) can be rewritten in the form:

$$\begin{cases} {}^{*}\nabla^{(\beta)}\upsilon = 0, \\ \rho\dfrac{\partial \upsilon}{\partial t} = -{}^{*}\nabla^{(\beta)}p\bar{\mathrm{I}} + 2\mu{}^{*}\nabla^{(\beta)}\mathbf{\Lambda} + \rho b - \rho[\upsilon^{*}\nabla^{(\beta)}]\upsilon, \\ \upsilon = \upsilon_0 \end{cases} \tag{13}$$

or

$$\begin{cases} {}^{*}\nabla^{(\beta)}\upsilon = 0, \\ \rho\dfrac{\partial \upsilon}{\partial t} = -{}^{*}\nabla^{(\beta)}p\bar{\mathrm{I}} + \mu{}^{*}\nabla^{(2\beta)}\upsilon + \rho b - \rho[\upsilon^{*}\nabla^{(\beta)}]\upsilon, \\ \upsilon = \upsilon_0 \end{cases} \tag{14}$$

In the case eq. (14) is called as the fractional Navier-Stoke equations.

Without the specific body force in eq. (14), we obtain a new form of the fractional Navier-Stoke equations:

$$\begin{cases} {}^{*}\nabla^{(\beta)}\upsilon = 0, \\ \rho\dfrac{\partial \upsilon}{\partial t} = -{}^{*}\nabla^{(\beta)} p\bar{\mathrm{I}} + \mu\,{}^{*}\nabla^{(2\beta)}\upsilon - \rho[\upsilon\,{}^{*}\nabla^{(\beta)}]\upsilon, \\ \upsilon = \upsilon_0 \end{cases} \tag{15}$$

Conclusion

In our work, we used the fractional gradient and Laplacian operators via fractional derivatives without singular kernel to investigate the mathematical theory for the heat and fluid flows. The fractional heat-conduction equations and the fractional Navier-Stokes equations were discussed. The results have opened the new directions of the heat and fluid flows within fractional derivatives without singular kernel.

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Nomenclature

b – specific body force, [Nm^{-3}]
c – specific heat of the material, [$Jkg^{-1}K^{-1}$]
p – thermodynamic pressure, [Pam^{-3}]
t – time, [s]

Greek symbols

β – fractional order, [–]
κ – thermal conductivity, [WK^{-1}]
υ – fluid velocity [ms^{-1}]
ρ – fluid density, [kgm^{-3}]
${}^{*}\nabla^{(\beta)}$ – fractional gradient operator, [m^{-1}]
${}^{*}\nabla^{(2\beta)}$ – fractional Laplacian operator, [m^{-1}]

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