LOCAL FRACTIONAL EULER'S METHOD FOR THE STEADY HEAT-CONDUCTION PROBLEM

by

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In this paper, the local fractional Euler's method is proposed to consider the steady heat-conduction problem for the first time. The numerical solution for the local fractional heat-relaxation equation is presented. The comparison between numerical and exact solutions is discussed.

Key words: *heat-conduction problem, heat-relaxation equation, numerical solution, local fractional Euler's method, local fractional derivative*

Introduction

The theory of local fractional calculus (LFC) [1-8] has been a potential tool for solving the non-differentiable problems in applied science, especially in thermal problems defined on Cantor sets [3, 9, 10]. The ordinary differential equations (ODE) via local fractional derivative (LFD) were reported to model the steady phenomena in engineering practice [3, 9-11]. In order to solve the problems, there are a lot of the proposed methods for the ODE, such as the local fractional Fourier transform [1], local fractional Laplace transform [1], local fractional Sumudu transform [10], and the asymptotic perturbation method [11]. The numerical integration via the LFC was proposed in [12].

However, the numerical differentiation via the LFC have not developed. In this article, our aim is to propose the local fractional Euler's method to solve the steady heat-conduction problem (local fractional heat-relaxation equation), which is given by the expression [9]:

$$-k\frac{\mathrm{d}^{\vartheta}\Lambda(\mu)}{\mathrm{d}\mu^{\vartheta}}=\Theta(\mu) \tag{1}$$

where k represents the thermal conductivity of the fractal material and $\Theta(\mu)$ is the fractal heat flow.

Mathematical tools and local fractional Euler's method

In this section the basic theory of the LFD is given and the local fractional Euler's method used in this paper is introduced.

Let $\mathbb{C}_{\vartheta}(a,b)$ be a set of the non-differentiable functions [1-12].

The LFD of $\Lambda(\mu)$ of order ϑ $(0<\vartheta<1)$ at the point $\mu=\mu_0$ is given, [1-12]:

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$$D^{(\vartheta)}\Lambda(\mu_0)=\frac{d^{\vartheta}\Lambda(\mu_0)}{d\mu^{\vartheta}}=\frac{\Delta^{\vartheta}[\Lambda(\mu)-\Lambda(\mu_0)]}{(\mu-\mu_0)^{\vartheta}} \tag{2}$$

where

$$\Delta^{\vartheta}[\Lambda(\mu)-\Lambda(\mu_0)]\cong\Gamma(1+\vartheta)[\Lambda(\mu)-\Lambda(\mu_0)] \tag{3}$$

with $\Lambda(\mu)\in\mathbb{C}_{\vartheta}(a,b)$.

Local fractional Taylor's theorem is given as [1, 8, 9, 12]:

Suppose that $\Lambda^{(k+1)\vartheta}(\mu)\in\mathbb{C}_{\vartheta}(a,b)$, for $k=0,1,...,n$ and $0<\vartheta\le 1$, then we have:

$$\Lambda(\mu)=\sum_{k=0}^{n}\frac{\Lambda^{(k\vartheta)}(\mu_0)}{\Gamma(1+k\vartheta)}(\mu-\mu_0)^{k\vartheta}+R_{n\vartheta}(\mu-\mu_0) \tag{4}$$

where $a<\mu_0<\xi<\mu<b,\ \forall\mu\in(a,b),\ \Lambda^{[(k+1)\vartheta]}(\mu)=\overbrace{D_{\mu}^{(\vartheta)}...D_{\mu}^{(\vartheta)}}^{k+1-times}\Lambda(\mu)$ and $R_{n\vartheta}(\mu-\mu_0)=$ $=O[(\mu-\mu_0)^{n\vartheta}]$.

In view of eq. (4), we have:

$$\Lambda(\mu)=\Lambda(\mu_0)+\frac{\Lambda^{(\vartheta)}(\mu_0)}{\Gamma(1+\vartheta)}(\mu-\mu_0)^{\vartheta}+R_{\vartheta}(\mu-\mu_0) \tag{5}$$

where $R_{\vartheta}(\mu-\mu_0)=O[(\mu-\mu_0)^{\vartheta}]$.

Let $\mu=\tau+\varsigma$, then we obtain:

$$\Lambda(\tau+\varsigma)\approx\Lambda(\tau)+\frac{\Lambda^{(\vartheta)}(\tau)}{\Gamma(1+\vartheta)}\varsigma^{\vartheta} \tag{6}$$

In order to present the numerical method, we can write $\tau_n=n\varsigma$ and $\Lambda_n=\Lambda(\tau_n)$.

The approximation in eq. (6) gives rise to the method of numerical solution:

$$\Lambda_{n+1}=\Lambda_n+\frac{\varsigma^{\vartheta}}{\Gamma(1+\vartheta)}\Xi(\Lambda_n,\tau_n) \tag{7}$$

where $[d^{\vartheta}\Lambda(\tau)]/d\tau^{\vartheta}=\Xi[\Lambda(\tau),\tau]$ and $\Lambda_0=\Lambda(0)$.

Thus, taking in succession $n=0,1,...,n-1$, the approximate values Λ_n at the mesh points τ_n can be easily obtained. This numerical method is known as the *local fractional Euler's method.*

When $\vartheta=1$, eq. (7) becomes the classical *Euler's method* [13].

The numerical solution for the steady heat-conduction problem

We now rewrite the local fractional heat-relaxation equation [9]:

$$\frac{d^{\vartheta}\Lambda(\mu)}{d\mu^{\vartheta}}=-\frac{1}{k}\Lambda(\mu) \tag{8a}$$

subject to the initial value condition:

$$\Lambda(0)=1 \tag{8b}$$

where the exact solution is given as:

$$\Lambda(\mu)=E_{\vartheta}\left(-\frac{1}{k}\mu^{\vartheta}\right) \tag{8c}$$

and k represents the thermal conductivity of the fractal material.

Suppose that we apply a timestep ς then $\mu_n = n\varsigma$, $\Lambda_n = \Lambda(\mu_n)$, and the local fractional Euler's method gives:

$$\Lambda_{n+1}=\Lambda_n-\frac{\varsigma^{\vartheta}}{k\Gamma(1+\vartheta)}\Lambda_n=\left[1-\frac{\varsigma^{\vartheta}}{k\Gamma(1+\vartheta)}\right]\Lambda_n \tag{9}$$

with $\Lambda_0 = \Lambda(0) = 1$.

Thus, we have:

$$\Lambda_1=\left[1-\frac{\varsigma^{\vartheta}}{k\Gamma(1+\vartheta)}\right] \tag{10a}$$

$$\Lambda_2=\left[1-\frac{\varsigma^{\vartheta}}{k\Gamma(1+\vartheta)}\right]^2 \tag{10b}$$

$$\Lambda_3=\left[1-\frac{\varsigma^{\vartheta}}{k\Gamma(1+\vartheta)}\right]^3 \tag{10c}$$

and so in general:

$$\Lambda_n=\left[1-\frac{\varsigma^{\vartheta}}{k\Gamma(1+\vartheta)}\right]^n \tag{10d}$$

The approximate values of the local fractional heat-relaxation equation when $\vartheta = \ln 2/\ln 3$ and $k = 1$ and are listed in tab. 1.

Table 1. The numerical approximation of the solution of non-differentiable type

n	ς	Λ_n
0	1	1
1	1	$1-1^{\ln 2/\ln 3}/\Gamma(1+\ln 2/\ln 3)$
2	1	$[1-1^{\ln 2/\ln 3}/\Gamma(1+\ln 2/\ln 3)]^2$
3	1	$[1-1^{\ln 2/\ln 3}/\Gamma(1+\ln 2/\ln 3)]^3$
4	1	$[1-1^{\ln 2/\ln 3}/\Gamma(1+\ln 2/\ln 3)]^4$

Suppose that the timestep ς is refined, and then our numerical approximation tends to the exact solution, namely:

$$\Lambda(\mu)\approx\left[1-\frac{\left(\frac{\mu}{n}\right)^{\vartheta}}{k\Gamma(1+\vartheta)}\right]^{\frac{\mu}{\varsigma}} \tag{11a}$$

$$\Lambda(\mu)=\lim_{n\to\infty}\left[1-\frac{\left(\frac{\mu}{n}\right)^{\vartheta}}{k\Gamma(1+\vartheta)}\right]^n=E_{\vartheta}\left(-\frac{1}{k}\mu^{\vartheta}\right) \tag{11b}$$

The numerical and exact solutions of the local fractional heat-relaxation equation with the smaller value $\varsigma = 1$ when Λ_1, Λ_2, Λ_3, Λ_4 and $\Lambda(\mu)$ are shown in fig. 1.

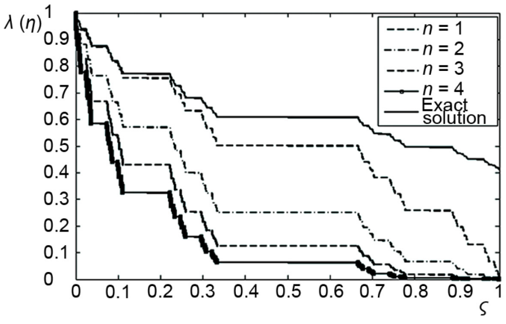

Figure 1. The numerical and exact solutions with the smaller value $\varsigma = 1$

Conclusions

In this work, we proposed the local fractional Euler's method to consider the steady heat-conduction problem. We presented the proposed method to obtain the numerical approximation of the non-differentiable solution for the local fractional heat-relaxation equation. The comparison between numerical and exact solutions of the steady heat-conduction problem was discussed.

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Nomenclature

μ – space co-ordinate, [m]

ϑ – fractal dimension, [–]

$\Lambda(\mu)$ – temperature, [K]

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