THE INTEGRATING FACTOR METHOD FOR SOLVING THE STEADY HEAT TRANSFER PROBLEMS IN FRACTAL MEDIA

by

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In this paper, we propose the integrating factor method via local fractional derivative for the first time. We use the proposed method to handle the steady heat-transfer equations in fractal media with the constant coefficients. Finally, we discuss the non-differentiable behaviors of fractal heat-transfer problems.

Key words: *fractal heat transfer, integrating factor method, exact solution, fractals, local fractional derivative*

Introduction

Local fractional differential equations [1] were used to model the non-differentiable problems, such as heat transfers [2-4], population dynamics [5], damped vibrations [6], and other [7-11]. Local fractional ordinary differential equations (ODE) were considered for the descriptions of the steady heat transfers (SHT) [4], growths of populations [5] and linear oscillator in vibrations [6]. Some technologies for local ODE, such as asymptotic perturbation method [6], local fractional Sumudu transforms [12], local fractional Laplace transforms [13], and local fractional Fourier transforms [14].

We consider the SHT equations in fractal media [15]:

$$\nu \frac{\mathrm{d}^{\vartheta} \varpi(\mu)}{\mathrm{d}\mu^{\vartheta}} = \omega(\mu) \tag{1a}$$

where ν is the fractal thermal conductivity, $\varpi(\mu)$ – the fractal temperature, $\omega(\mu)$ – the fractal heat flux, and the local fractional operator of $\Lambda(\mu)$ of the order $\vartheta\,(0 < \vartheta < 1)$ is defined by [1-15]:

$$\frac{\mathrm{d}^{\vartheta} \Lambda(\mu)}{\mathrm{d}\mu^{\vartheta}}\Big|_{\mu=\mu_0} = \lim_{\mu \to \mu_0} \frac{\Delta^{\vartheta}[\Lambda(\mu) - \Lambda(\mu_0)]}{(\mu - \mu_0)^{\vartheta}} \tag{1b}$$

with

$$\Delta^{\vartheta}[\Lambda(\mu) - \Lambda(\mu_0)] \cong \Gamma(1+\vartheta)[\Lambda(\mu) - \Lambda(\mu_0)] \tag{1c}$$

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When the fractal heat flux takes the form:

$$\omega(\mu)=\psi(\mu)\varpi(\mu)+\phi(\mu) \tag{2a}$$

the SHT equations in fractal media is rewritten:

$$\nu\frac{\mathrm{d}^{\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{\vartheta}}=\psi(\mu)\varpi(\mu)+\phi(\mu) \tag{2b}$$

which leads to:

$$\frac{\mathrm{d}^{\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{\vartheta}}=\frac{\psi(\mu)}{\nu}\varpi(\mu)+\frac{\phi(\mu)}{\nu} \tag{2b}$$

In this article, we consider the SHT problem in fractal media:

$$\frac{\mathrm{d}^{\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{\vartheta}}=\theta(\mu)\varpi(\mu)+\varphi(\mu) \tag{3}$$

where $\theta(\mu)=\psi(\mu)/\nu=\theta\neq 0$ and $\varphi(\mu)=\phi(\mu)/\nu=\varphi$. The main aims of this article are to present the integrating factor method (IFM) to solve the SHT problem in fractal media.

Mathematical theory

Let us consider the linear local fractional ODE with constant coefficients:

$$\frac{\mathrm{d}^{\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{\vartheta}}=\theta\varpi(\mu)+\varphi \tag{4}$$

where θ and φ are constant coefficients and $\theta\neq 0$.

Theorem 1. For the constants $\theta,\ \varphi\in\mathbb{R}$ with $\theta\neq 0,$ the linear local fractional ODE:

$$\frac{\mathrm{d}^{\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{\vartheta}}=-\theta\varpi(\mu)+\varphi \tag{5}$$

has infinitely many solutions, one for each value of $K\in\mathbb{R},$ given by:

$$\varpi(\mu)=KE_{\vartheta}(-\theta\mu^{\vartheta})+\frac{\varphi}{\theta} \tag{6}$$

where $E_{\vartheta}(\mu^{\vartheta})=\Sigma_{i=0}^{\infty}\mu^{i\vartheta}/\Gamma(1+i\vartheta)$.

Proof. By using the relationship [1, 15]:

$$\frac{\mathrm{d}^{\vartheta}[\varpi(\mu)\upsilon(\mu)]}{\mathrm{d}\mu^{\vartheta}}=\upsilon(\mu)\frac{\mathrm{d}^{\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{\vartheta}}+\varpi(\mu)\frac{\mathrm{d}^{\vartheta}\upsilon(\mu)}{\mathrm{d}\mu^{\vartheta}} \tag{7}$$

we have:

$$E_{\vartheta}(\theta\mu^{\vartheta})\left[\frac{\mathrm{d}^{\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{\vartheta}}+\theta\varpi(\mu)\right]=\frac{\mathrm{d}^{\vartheta}[\varpi(\mu)E_{\vartheta}(\theta\mu^{\vartheta})]}{\mathrm{d}\mu^{\vartheta}} \tag{8}$$

such that:

$$\frac{\mathrm{d}^{\vartheta}[\varpi(\mu)E_{\vartheta}(\theta\mu^{\vartheta})]}{\mathrm{d}\mu^{\vartheta}} = \varphi E_{\vartheta}(\theta\mu^{\vartheta}) \tag{9}$$

which leads to:

$$\begin{aligned}\varpi(\mu)E_{\vartheta}(\theta\mu^{\vartheta}) &= \frac{1}{\Gamma(1+\vartheta)}\int \varphi E_{\vartheta}(\theta\mu^{\vartheta})(\mathrm{d}\mu)^{\vartheta} \\ &= \frac{\varphi}{\theta}E_{\vartheta}(\theta\mu^{\vartheta}) + K\end{aligned} \tag{10}$$

where K is an any constant and the local fractional indefinite integral of $\Pi(\mu)$, denoted as $\Theta(\mu)$, is defined:

$$\Theta(\mu) = \frac{1}{\Gamma(1+\vartheta)}\int \Pi(\mu)(\mathrm{d}\mu)^{\vartheta} \tag{11}$$

Thus, we obtain:

$$\varpi(\mu) = KE_{\vartheta}(-\theta\mu^{\vartheta}) + \frac{\varphi}{\theta} \tag{12}$$

Therefore, we finish the proof.

We say that the function $\Xi(\mu) = E_{\vartheta}(\theta\mu^{\vartheta})$ is an integrating factor function. This method is called as the IFM.

Solving SHT equations in fractal media with the constant coefficients

In this section, we consider the two illustrative examples for SHT problems with the constant coefficients.

Example 1

Let us consider the following SHT equation with the constant coefficients:

$$\frac{\mathrm{d}^{\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{\vartheta}} = \varpi(\mu) \tag{13a}$$

subject to the initial value condition:

$$\varpi(0) = 2 \tag{13b}$$

In view of eqs. (5) and (6), we have:

$$\theta = -1 \quad \text{and} \quad \varphi = 0 \tag{13c,d}$$

such that:

$$\varpi(\mu) = 2E_{\vartheta}(\mu^{\vartheta}) \tag{13e}$$

and its corresponding graph is shown in fig. 1.

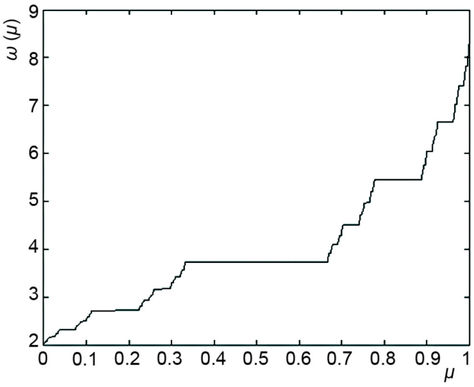

Figure 1. The plot of $\varpi(\mu)$ when ϑ = ln2/ln3

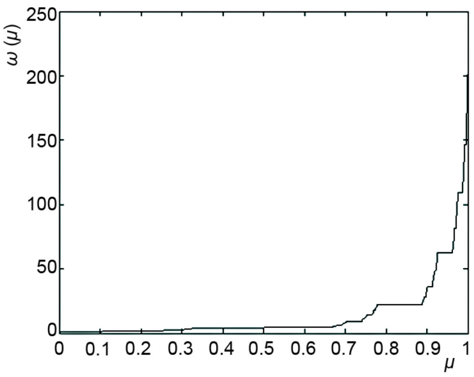

Figure 2. The plot of $\varpi(\mu)$ when ϑ = ln2/ln3

Example 2

We consider the SHT equation with the constant coefficients:

$$\frac{d^{\vartheta}\varpi(\mu)}{d\mu^{\vartheta}} = 3\varpi(\mu) + 2 \tag{14a}$$

subject to the initial value condition:

$$\varpi(0) = 1 \tag{14b}$$

Making use of eqs. (5) and (6), we have:

$$\varpi(\mu) = \frac{1}{3}E_{\vartheta}(3\mu^{\vartheta}) + \frac{2}{3} \tag{14c}$$

and its corresponding graph is shown in fig. 2.

Conclusions

In this work, we proposed the IFT for solving the SHT equation with the constant coefficients via local fractional derivative. The exaction solutions for the fractal SHT equations were discussed. The non-differentiable graphs showed the fractal behaviors of the heat-transfer problems described by local fractional ordinary differential equations.

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Nomenclature

μ – space co-ordinate, [m]
ϑ – fractal order, [–]
$\varpi(\mu)$ – temperature, [K]

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