ON STEADY HEAT FLOW PROBLEM INVOLVING YANG-SRIVASTAVA-MACHADO FRACTIONAL DERIVATIVE WITHOUT SINGULAR KERNEL

by

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In this article, we present a new application for the Yang-Srivastava-Machado fractional derivative without singular kernel to the steady heat flow problem. The Sumudu transform is used to find the analytical solution of the fractional-order heat flow.

Key words: *steady heat conduction, heat flow, analytical solution, Sumudu transform, Yang-Srivastava-Machado fractional derivative*

Introduction

Fractional calculus has been important applications in applied science [1-5]. We recall the definitions of the fractional derivatives.

The Riemann-Liouville fractional derivative of the function $\Omega(x)$ of fractional order ν is defined by [5, 6]:

$$\mathrm{D}_{a^+}^{(\nu)}\Omega(x) = \frac{1}{\Gamma(n-\nu)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x \frac{\Omega(\lambda)}{(x-\lambda)^{\nu+1-n}}\mathrm{d}\lambda \tag{1}$$

where $a \le x$, n is an integer, and ν is a real number.

The Caputo fractional derivative of the function $T(x)$ of fractional order ν is defined by [5, 6]:

$$\mathrm{D}_{a^+}^{(\nu)}\Omega(x) = \frac{1}{\Gamma(n-\nu)}\int_a^x \frac{1}{(x-\lambda)^{\nu+1-n}}\left(\frac{\mathrm{d}^n\Omega(\lambda)}{\mathrm{d}\lambda^n}\right)\mathrm{d}\lambda \tag{2}$$

Recently, following eq. (2), Caputo and Fabrizio proposed the fractional derivative operator without singular kernel, which was given [7-9]:

$$^{CF}\mathrm{D}_x^{(\nu)}\Omega(x) = \frac{(2-\nu)\Im(\nu)}{2(1-\nu)}\int_0^x \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right]\Omega^{(1)}(\lambda)\mathrm{d}\lambda \tag{3}$$

where $\Im(\nu)$ is a normalization constant depending on ν $(0<\nu<1)$.

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Losada and Nieto proposed that for $\Im(\nu) = 2/(2-\nu)$, eq. (3) could lead to the new Caputo-Fabrizio fractional derivative operator given [7]:

$$ {}_{*}^{CF}\mathrm{D}_{x}^{(\nu)}\Omega(x) = \frac{1}{1-\nu}\int_{0}^{x} \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right]\Omega^{(1)}(\lambda)\mathrm{d}\lambda \tag{4} $$

where ν $(0 < \nu < 1)$ is a real number.

More recently, in view of eq. (1), Yang, Srivastava, and Machado proposed a new fractional derivative without singular kernel, given by the expression [9]:

$$ {}^{YSM}\mathrm{D}_{a^+}^{(\nu)}\Omega(x) = \frac{\Re(\nu)}{1-\nu}\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x} \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right]\Omega(\lambda)\mathrm{d}\lambda \tag{5} $$

where $a \le x,$ ν $(0 < \nu < 1)$ is a real number, and $\Re(\nu)$ is a normalization function depending on ν such that $\Re(0) = \Re(1) = 1$.

Let $0 < \nu < 1$ and $\Re(\nu) = 1,$ then eq. (5) can be written [9]:

$$ {}_{*}^{YSM}\mathrm{D}_{a^+}^{(\nu)}\Omega(x) = \frac{1}{1-\nu}\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x} \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right]\Omega(\lambda)\mathrm{d}\lambda \tag{6} $$

Equation (5) is called as the Yang-Srivastava-Machado fractional derivative.

Based on eq. (5), the fractional-order Fourier law in 1-D case was written in the form [9]:

$$ K\,{}^{YSM}\mathrm{D}_{0}^{(\nu)}\psi(x) = -H(x) \tag{7} $$

where K is the thermal conductivity of the material, $\psi(x)$ – the temperature, and $H(x)$ – the heat flow.

The main aim of the article is to present the new application of the Yang-Srivastava-Machado fractional derivative without singular kernel to the steady heat flow and find the analytic solution of the steady heat-conduction of fractional order by the Sumudu transform [10-13].

Mathematical method

With the help of the approximation to the identity [14]:

$$ \delta(x-\lambda) = \lim_{\psi\to 0}\frac{1}{\psi}\exp\left(-\frac{x-\lambda}{\psi}\right) \tag{8} $$

where $\nu \to 1,$ eq. (5) becomes:

$$ \lim_{\nu\to 1} {}^{YSM}\mathrm{D}_{a^+}^{(\nu)}\Omega(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x}\delta(x-\lambda)\Omega(\lambda)\mathrm{d}\lambda = \Omega^{(1)}(x) \tag{9} $$

When $\nu \to 0,$ eq. (5) becomes:

$$ \lim_{\nu\to 0} {}^{YSM}\mathrm{D}_{a^+}^{(\nu)}\Omega(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x}\Omega(\lambda)\mathrm{d}\lambda = \Omega(x) \tag{10} $$

Taking the Sumudu transform of the Yang-Srivastava-Machado fractional derivative without singular kernel for the parameter $a = 0$, we have:

$$\begin{aligned}
&\wp\left({}^{YSM}\mathrm{D}_0^{(\nu)}\Omega(x)\right) = \\
&= \frac{1}{\varpi}\int_0^x \exp\left(-\frac{x}{\varpi}\right)\left\{\frac{\Re(\nu)}{1-\nu}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right]\Omega(\lambda)\mathrm{d}\lambda\right\}\mathrm{d}x \\
&= \frac{\Re(\nu)}{1-\nu}\frac{\wp\left(\int_0^t \exp\left[-\frac{\nu}{1-\nu}(t-\lambda)\right]\Omega(\lambda)\mathrm{d}\lambda\right)}{\varpi} \\
&= \frac{\Re(\nu)\Omega(\varpi)}{1-\nu}\wp\left[\exp\left(-\frac{\nu}{1-\nu}t\right)\right] \\
&= \frac{\Re(\nu)\Omega(\varpi)}{(1-\nu)+\nu\varpi}
\end{aligned} \tag{11}$$

where

$$\wp[\xi(x)] := \frac{1}{\varpi}\int_0^x \exp\left(-\frac{x}{\varpi}\right)\xi(x)\mathrm{d}x = \xi(\varpi) \tag{12}$$

denotes the Sumudu transform of $\xi(x)$ [10-13].

Solving the steady heat flow via Yang-Srivastava-Machado fractional derivative without singular kernel

When $H(x) = \psi(x) + b$, where b is a constant, eq. (7) is the fractional heat relaxation equation in the steady heat-conduction problem:

$$K\,{}^{YSM}\mathrm{D}_0^{(\nu)}\psi(x) = -\psi(x) - b \tag{13}$$

with the initial value condition $\psi(0) = 1$, where K is the thermal conductivity of the material.

By taking the Sumudu transform of eq. (13), eq. (13) can be written in the form:

$$K\frac{\Re(\nu)\psi(\varpi)}{(1-\nu)+\nu\varpi} = -\psi(\varpi) - b \tag{14}$$

which leads to:

$$\psi(\varpi) = \frac{-b[(1-\nu)+\nu\varpi]}{K\Re(\nu)+(1-\nu)+\nu\varpi} \tag{15}$$

Thus, eq. (15) becomes:

$$\psi(\varpi) = \frac{\dfrac{b[K\Re(\nu)-1]}{K\Re(\nu)+(1-\nu)}}{1+\dfrac{\nu}{K\Re(\nu)+(1-\nu)}\varpi} \tag{16}$$

Therefore, we have:

$$\psi(x) = \frac{b[K\Re(\nu)-1]}{K\Re(\nu)+(1-\nu)}\exp\left[-\frac{\nu x}{K\Re(\nu)+(1-\nu)}\right] \tag{17}$$

and its corresponding graph at different values of $\Re(\nu)=1$, $K=2$, $b=1$, $\nu=0.5$, 0.7, 0.9, and 1 is shown in fig. 1.

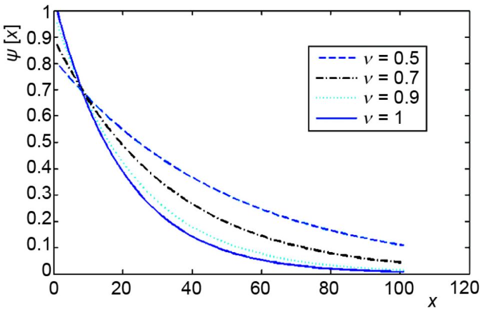

Figure 1. Chat of solutions at different values of ν = 0.5, 0.7, 0.9, and 1 for $\psi(x)$

Conclusions

In this work, we investigate the new application for the YSM fractional derivative without singular kernel to the steady heat flow of fractional order. The Sumudu transform was used to solve the steady heat flow of fractional order. The obtained solution was given to show the behavior of the steady heat flow of fractional order.

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Nomenclature

x – space co-ordinate, [m]

Greek symbols

ν – fractional order, [–]
$\wp[\xi(x)]$ – Sumudu transform, [–]

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