A NEW NUMERICAL METHOD FOR SOLVING TWO-DIMENSIONAL VARIABLE-ORDER ANOMALOUS SUB-DIFFUSION EQUATION

by

Wei JIANG and Beibei GUO *

Department of Mathematics, Harbin Institute of Technology at Weihai, Shandong, China

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The novelty and innovativeness of this paper are the combination of reproducing kernel theory and spline, this leads to a new simple but effective numerical method for solving variable-order anomalous sub-diffusion equation successfully. This combination overcomes the weaknesses of piecewise polynomials that can not be used to solve differential equations directly because of lack of the smoothness. Moreover, new bases of reproducing kernel spaces are constructed. On the other hand, the existence of any ε-approximate solution is proved and an effective method for obtaining the ε-approximate solution is established. A numerical example is given to show the accuracy and effectiveness of theoretical results.

Key words: *variable-order anomalous sub-diffusion equation, spline, reproducing kernel theory, approximate solution*

Introduction

Because of their practical applications, fractional calculus has received much attention in recent years. For example, fractal heat-transfer problems [1], local fractional Burger's equations [2], volterra fractional equations [3,4], discrete fractional equations [5, 6], and others [7, 8]. In order to more accurately describe the evolution of a system, the concept of a variable-order operator has been developed [9]. Sun [10] proposed variable-order fractional differential operators in anomalous diffusion modeling. The numerical solutions of variable-order fractional diffusion equations have also been considered in some papers [11-14]. The field is still at an early stage of development. In view of their complexity and usefulness, it is interesting to develop high accuracy numerical methods for variable-order fractional differential equations.

In this paper, we will investigate numerical methods of the 2-D variable-order anomalous sub-diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = \mathrm{D}_t^{1-\gamma(x,t)} \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \tag{1}$$

subject to initial and boundary conditions:

$$u(x,0)=\phi(x), \qquad u(0,t)=\varphi(t), \qquad u(L,t)=\psi(t), \qquad 0 \le x \le L, \qquad 0 < t \le T \tag{2}$$

* Corresponding author; e-mail: guobei066@126.com

where $0 < \gamma_{\min} \le \gamma(x,\ t) \le \gamma_{\max} < 1$ and $\mathrm{D}_t^{1-\gamma(x,t)} v(x,t)$ is the variable-order Caputo fractional derivative of order $1-\gamma(x,t)$ defined as [10]:

$$\mathrm{D}_t^{1-\gamma(x,t)} v(x,t) = \frac{1}{\Gamma[\gamma(x,t)]} \int_0^t \frac{v'(x,\eta)}{(t-\eta)^{1-\gamma(x,t)}} \mathrm{d}\eta \tag{3}$$

Here we only consider homogeneous initial and boundary conditions $u(x,0)=0$, $u(L,t)=0$, $u(0,t)=0,$ since non-homogeneous initial and boundary conditions can be reduced to the homogeneous conditions easily.

The main contribution of this paper is that a new numerical method for solving eq. (1) with (2) is introduced by combining reproducing kernel theory and spline. Noting that piecewise polynomials are continuous but lack of the smoothness, they can not be used to solve differential equations directly. However, in this paper, by applying reproducing kernel method and some ideas of spline, new bases of reproducing kernel spaces are constructed. Our numerical algorithm with the new bases can reach the precision of spline, thus the effectiveness of numerical method is improved correspondingly. At the same time, the existence of any ε-approximate solution is proved and an effective method for obtaining the ε-approximate solution is established. Some numerical results are given. By comparing with [14], it can be seen this numerical method is more simple and effective. It is worth noting that this method can also be used to solving other problems.

Preliminaries

Let $\Omega = [0,L]\times[0,T]$, we now introduce several definitions of spaces which will be needed in this paper.

Definition 1. $W_2^n[0,T] = \{u(x) \mid u(x), u^{(i)}(x), i=0,1,\cdots,n-1, n\in\mathbb{N}$ are absolutely continuous real value function in $[0,T], u(0)=0, u^{(n)}(x)\in L^2[0,T]\}$. The inner product and norm in $W_2^n[0,T]$ are given, respectively, by:

$$\langle u(x), v(x)\rangle_{W_2^n} = \sum_{i=0}^{n} u^{(i)}(0)v^{(i)}(0) + \int_0^T u^{(n+1)}(x)v^{(n+1)}(x)\mathrm{d}x, \quad \|u\|_{W_2^n} = \langle u(x), v(x)\rangle_{W_2^n}^{\frac{1}{2}} \tag{4}$$

Similar to [15], we can prove that $W_2^n[0,T]$ is a reproducing kernel space. That is, for any fixed $x\in[0,T]$, there exists a $r_n(x,y)\in W_2^n[0,T]$, such that for any $u(x)\in W_2^n[0,T]$, the reproducing kernel $r_n(x,y)$ can be denoted by:

$$u(x) = \langle u(y), \qquad r_n(x,y)\rangle_{W_2^n} \tag{5}$$

Definition 2. $\bar{W}_2^m[0,L] = \{u(x) \mid u(x), u^{(i)}(x), i=0,1,\cdots,m-1, m\in\mathbb{N}$ are absolutely continuous real value function in $[0,L], u(0)=0, u(L)=0, u^{(n)}(x)\in L^2[0,L]\}$. The inner product and norm in $\bar{W}_2^m[0,L]$ are given, respectively, by:

$$\langle u(x), v(x)\rangle_{\bar{W}_2^m} = \sum_{i=0}^{m} u^{(i)}(0)v^{(i)}(0) + \int_0^L u^{(m+1)}(x)v^{(m+1)}(x)\mathrm{d}x, \quad \|u\|_{\bar{W}_2^m} = \langle u(x), u(x)\rangle_{\bar{W}_2^m}^{\frac{1}{2}} \tag{6}$$

Analogously, it can prove that $\bar{W}_2^m[0,L]$ is also a reproducing kernel space, and the reproducing kernel $R_m(x,y)$ can be denoted by:

$$u(x) = \langle u(y), \qquad R_m(x, y)\rangle_{\bar{W}_2^m} \tag{7}$$

Next, the expressions of some reproducing kernels are given, which will be used in the following. The reproducing kernel of $W_2^2[0, T]$, $\bar{W}_2^1[0, L]$ is given, respectively, by:

$$r_2(x, y) = \begin{cases} xy + \dfrac{xy^2}{2} - \dfrac{y^3}{6}, & y \le x, \\ xy + \dfrac{yx^2}{2} - \dfrac{x^3}{6}, & x < y. \end{cases}, \qquad R_1(x, y) = \begin{cases} 1 + y, \ y \le x, \\ 1 + x, \ x < y. \end{cases}$$

Define $W(\Omega) \triangleq \bar{W}_2^3[0, L] \otimes W_2^2[0, T]$, and $W(\Omega)$ is a reproducing kernel space with reproducing kernel $R_{3,2}(x, y, t, s) = R_3(x, y) \cdot r_2(t, s)$, where the direct product $\otimes$ is defined in [16], $R_3(x, y)$, $r_2(t, s)$ is the reproducing kernel of $\bar{W}_2^3[0, L]$, $W_2^2[0, T]$, respectively. Similar to $W(\Omega)$, we can define $W_0(\Omega) \triangleq \bar{W}_2^1[0, L] \otimes W_2^1[0, T]$, and it is a reproducing kernel spaces with the reproducing kernel $R_{1,1}(x, y, t, s) = R_1(x, y) \cdot r_1(t, s)$, where $r_1(t, s)$ is the reproducing kernel of $W_1^2[0, T]$.

Consider a partition of $[0,d]$, $\pi : 0 = h_0 < h_1 < \cdots h_{n-1} < h_n = d$. Let $J_j = [h_{j-1}, h_j]$, $j = 1, 2, \cdots, n$, then the cubic spline space $S_{3,\pi}$ is defined by:

$$S_{3,\pi} = \{u \in C^2[0, d] : u\big|_{J_j} \in p_3, \qquad j = 1, 2, 3, \cdots, n\} \tag{8}$$

where p_3 denotes the set of all polynomials with order not more than three over J_j. Meanwhile, we define $S_{3,\pi}^1 = \{u \in S_{3,\pi} : u(0) = 0\}$, $S_{3,\pi}^2 = \{u \in S_{3,\pi} : u(0) = u(L) = 0\}$.

A new base of $W(\Omega)$

In the previous section, we have given the reproducing kernel $r_2(t, s)$ of $W_2^2[0, T]$, now, we introduce the new bases of $W_1(\Omega)$ and $W(\Omega)$. Assuming $\{x_i\}_{i=0}^{\infty}$ is a dense subset of $[0, L]$ with $x_0 = 0$, $x_1 = L$, and $\{t_i\}_{i=0}^{\infty}$ is a dense subset of $[0, T]$ with $t_0 = 0$, $t_1 = T$.

Remark 1. Note that $T = \{r_2(t, t_1), r_2(t, t_2), \cdots, r_2(t, t_n), \cdots\} \subset S_{3,\pi}^1$, but, T can not form a base of $S_{3,\pi}^1$ over the region $[0, T]$. So, The algorithms with T as a base can not reach the precision of spline.

But the following lemmas show that after adding two functions t, t^2 into T, the new T will be a base of $S_{3,\pi}^1$. Thus, the algorithms with the new T as a base can reach the precision of spline, the validity of numerical method is improved correspondingly.

Lemma 1. $\{t, t^2, r_2(t, t_1), r_2(t, t_2), \cdots, r_2(t, t_n), \cdots\}$ is linearly independent.

Proof: Take any integer $n \ge 2$, suppose $at + bt^2 + \Sigma_{i=1}^n c_i r_2(t, t_i) = 0$. For every $k \ge 2$, select $p \in W_2^2[0, T]$ such that $p(t_k) = 1$ and $p(t_j) = 0$, $j \ne k$. Meanwhile, $p(t) = 0$ when t is close to 0, T. Hence, $p'(0) = p'(T) = 0$. From eq. (4), it follows that:

$$0 = \langle at + bt^2 + \sum_{i=1}^{n} c_i r_2(t, t_i), \ p(t)\rangle = 0 + 0 + \sum_{i=1}^{n} c_i p(t_i) = c_k, \qquad k = 2, 3, \cdots, n$$

Hence, $at + bt^2 + c_1 r_2(t, T) = at + bt^2 + c_1(Tt + Tt^2/2 - t^3/6) = 0$, which gives $a = b = c_1 = 0$. This completes our proof.

Theorem 1. $\bar{K}_1 = \{t, t^2, r_2(t, t_1), r_2(t, t_2), \cdots, r_2(t, t_n)\}$ is a base of $S_{3,\pi}^1$.

Proof: Since $r_2(t, s)$ is the reproducing kernel of $W_2^2[0, T]$, $r_2(t, t_i) \in C^2[0, T]$. On the other hand, $r_2(t, t_i)$ is a piecewise cubic polynomial. So, $r_2(t, t_i) \in S_{3,\pi}$ for every $i = 1, 2, \cdots, n$. Noting that $t, t^2 \in S_{3,\pi}$, $\overline{K}_1 \in S_{3,\pi}^1$, since dim $S_{3,\pi}^1 = n + 2$ and $\overline{K}_1$ is a set whose element is linear independent, $\overline{K}_1$ is a base of $S_{3,\pi}^1$.

Theorem 2. $K_1 = \{t, t^2, r_2(t, t_1), r_2(t, t_2), \cdots, r_2(t, t_n), \cdots\}$ is a base of $W_2^2[0, T]$.

Proof: Since $\{r_2(t, t_1), r_2(t, t_2), \cdots, r_2(t, t_n), \cdots\}$ is complete in $W_2^2[0, T]$, So, K_1 is also complete in $W_2^2[0, T]$. Noting that K_1 is a set whose element is linearly independent, K_1 is a base of $W_2^2[0, T]$.

Putting $\varphi_i(x) = R_1(x, x_i)$, $i = 0, 1, 2, \cdots$, and:

$$G_i(x) = J_0^2 \varphi_i(x) = \int_0^x (x - v)\varphi_i(v)\mathrm{d}v, \qquad Q_i(x) = G_i(x) - \frac{xG_i(L)}{L} \tag{9}$$

then $Q_i(0) = Q_i(L) = 0$.

Lemma 2. $\{Q_0(x), Q_1(x), \cdots, Q_n(x), \cdots\}$ is linearly independent.

Proof: Take any positive integer n, suppose that $\Sigma_{i=0}^{\infty} c_i Q_i(x) = 0$. The previous equation were differentiated twice to yield $0 = \Sigma_{i=0}^{n} c_i Q_i''(x) = \Sigma_{i=0}^{n} c_i \varphi_i(x)$. Since $\varphi_i(x)$ is linearly independent, $c_i = 0$, $i = 0, 1, \cdots, n$. This completes our proof.

Theorem 3. $T_1 = \{Q_0(x), Q_1(x), \cdots, Q_n(x)\}$ is a base of $S_{3,\pi}^2$, where π is a partition composed of $\{x_0, x_1, \cdots, x_n\}$.

Proof: From $Q_i''(x) = \varphi_i(x) \in C[0, L]$ and $Q_i(x)$ is a piecewise cubic polynomial, we conclude that $T_1 \subset S_{3,\pi}$. Since $Q_i(0) = Q_i(L) = 0$, $T_1 \subset S_{3,\pi}^2$. From Lemma 2, T_1 is a set whose element is linearly independent. Using the fact dim $S_{3,\pi}^2 = n + 1 =$ dim $T_1 = n + 1$. We obtain that T_1 is a base of $S_{3,\pi}^2$.

Remark 2. Theorem 3 shows that algorithms with T_1 as a base can reach the precision of spline.

Theorem 4. $\{Q_0(x), Q_1(x), \cdots\}$ is a base of $\overline{W}_2^3[0, L]$.

Proof: Take any $f \in \overline{W}_2^3[0, L]$, then $f'' \in \overline{W}_2^1[0, L]$. Since $\{\varphi_i\}_{i=0}^{\infty}$ is a base of $\overline{W}_2^1[0, L]$:

$$f'' = \sum_{i=1}^{\infty} c_i \varphi_i(x) \tag{10}$$

With twice-integration of (10) and using the fact that $f(0) = 0$, we obtain:

$$f(x) = f'(0)x + \int_0^x (x - t)f''(t)\mathrm{d}t = f'(0)x + \sum_{i=1}^{\infty} c_i \int_0^x (x - t)\varphi_i(t)\mathrm{d}t = f'(0)x + \sum_{i=1}^{\infty} c_i G_i(x) \tag{11}$$

Taking $x = L$ in (11), we get $0 = f'(0)L + \Sigma_{i=1}^{\infty} c_i G_i(L)$, which gives $f'(0) = -\Sigma_{i=1}^{\infty} c_i G_i(L)/L$. Thus,

$$f(x) = \sum_{i=1}^{\infty} c_i \left[G_i(x) - \frac{x}{L} G_i(L) \right] = \sum_{i=1}^{\infty} c_i Q_i(x) \tag{12}$$

Combining (12) and Lemma 2, it yields that $\{Q_0(x), Q_1(x), \cdots\}$ is a base of $\bar{W}_2^3[0, L]$. According to Theorems 3 and 4, we get a new base of $W(\Omega)$:

$$\{H_k(x, t)\}_{k=1}^{\infty} = \{Q_i(x)t\}_{i=0}^{\infty} \cup \{Q_i(x)t^2\}_{i=0}^{\infty} \cup \{Q_i(x)r_2(t, t_i)\}_{i=0}^{\infty} \tag{13}$$

A numerical scheme for solving eq. (1) with eq. (2)

Defining

$$\mathbb{L}: W(\Omega) \to W_0(\Omega), \qquad \mathbb{L}u(x, t) \triangleq \frac{\partial u(x, t)}{\partial t} - D_t^{1-\gamma(x,t)} \frac{\partial^2 u(x, t)}{\partial x^2} \tag{14}$$

then eq. (1) is equivalent to:

$$\mathbb{L}u(x, t) = F(x, t) \tag{15}$$

Assuming that the solution of eq. (15) exists and is unique. In the following, we prove that $\mathbb{L}$ is a linear bound operator.

Lemma 3. $\mathbb{L}: W(\Omega) \to W_0(\Omega)$ is a linear bound operator.

Proof: Obviously, $\mathbb{L}$ is a linear operator. Next, we prove that $\mathbb{L}$ is bounded.

It follows from $R_{3,2}(x, y, t, s) \in W(\Omega)$ is the reproducing kernel of $W(\Omega)$, $u(x, t) \in W(\Omega)$ and the reproducing property of reproducing kernel that:

$$u(x, t) = \langle u(y, s), R_{3,2}(x, y, t, s)\rangle_{W(\Omega)} = \langle u(y, s), R_3(x, y) r_2(t, s)\rangle_{W(\Omega)} \tag{16}$$

According to (15) and (16), we get:

$$\begin{aligned} \| \mathbb{L}u \|_{W_0(\Omega)} &= \| \langle u(y, s), R_3(x, y) \frac{\partial r_2(t, s)}{\partial t}\rangle_{W(\Omega)} - \langle u(y, s), \\ &\frac{\partial^2}{\partial x^2} R_3(x, y) D_t^{1-\gamma(x,t)} r_2(t, s)\rangle_{W(\Omega)} \|_{W_0(\Omega)} \le \| u \|_{W(\Omega)} \\ &\left\{ \| R_3(x, y) \frac{\partial r_2(t, s)}{\partial t} \|_{y,s,W(\Omega)} + \| D_t^{1-\gamma(x,t)} \left[\frac{\partial^2 R_3(x, y)}{\partial x^2} r_2(t, s) \right] \|_{y,s,W(\Omega)} \right\} \end{aligned} \tag{17}$$

By simple calculating, (17) becomes:

$$\begin{aligned} \| \mathbb{L}u \|_{W_0(\Omega)} \le \| u \|_{W(\Omega)} &\left\{ \sqrt{R_3(x, x)} \cdot \sqrt{\frac{\partial^2}{\partial t_1 \partial t_2} r_2(t_1, t_2)} \,|_{t_1=t_2=t} + \sqrt{\frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial x_2^2} R_3(x_1, x_2)} \,|_{x_1=x_2=x} \right. \\ &\left. \frac{1}{\Gamma^2[\gamma(x, t)]} \int_0^{t_2} \frac{1}{(t_2 - \sigma_2)^{1-\gamma(x,t)}} \int_0^{t_1} \frac{\left| \partial_{t_1} \partial_{t_2} r_2(\sigma_1, \sigma_2) \right|}{(t_1 - \sigma_1)^{1-\gamma(x,t)}} \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \,|_{\sigma_1=\sigma_2=t} \right\} \end{aligned}$$

In terms of $R_3(x, y) \in \bar{W}_2^3[0, L]$, $r_2(t, s) \in W_2^2[0, T]$ and the definition of $\bar{W}_2^3[0, L]$ and $W_2^2[0, T]$, it infers that:

$$\sqrt{R_3(x,x)},\qquad \sqrt{\partial^2/\partial t_1\partial t_2 r_2(t_1,t_2)}\,\Big|_{t_1=t_2=t}\,,\qquad \sqrt{\partial^2/\partial x_1^2\partial^2/\partial_2^2 R_3(x_1,x_2)}$$

and $\partial_{t_1}\partial_{t_2}r_2(\sigma_1,\sigma_2)$ are all bounded. Hence, $\mathbb{L}$ is bounded. The conclusion holds.

Definition 3. Let $u_n(x,t)\in W(\Omega)$. For any $\varepsilon>0$, if

$$\|F-\mathbb{L}u_n\|_{C(\Omega)}=\underset{(x,t)\in\Omega}{Sup}\left|F\,(x,t)-\mathbb{L}u_n(x,t)\right|<\varepsilon \tag{18}$$

then we say that $u_n(x,t)$ is an ε-approximate solution of eq. (15).

Theorem 5. For any $\varepsilon > 0$, there exists a positive integer N, such that for every $n > N$:

$$u_n(x,t)=\sum_{i=1}^{n}a_i^*H_i(x,t)$$

is an ε-approximate solution of eq. (15), $\{a_i^*\}_{i=1}^n$ satisfies:

$$\|F-\mathbb{L}u_n\|_{C(\Omega)}=\min_{a_i\in\mathbb{R}}\|F(x,t)-\mathbb{L}(\sum_{i=1}^{n}a_iH_i(x,t)\|_{C(\Omega)} \tag{19}$$

Proof: Denote by $u\in W(\Omega)$ the exact solution of eq. (15). Since $\{H_i(x,t)\}_{i=1}^{\infty}$ is a base of $W(\Omega)$, then $u(x,t)=\Sigma_{i=1}^{\infty}a_iH_i(x,t)$. So for any given $\varepsilon>0$, there exists a positive integer N such that for every $n > N,$ there exists a function $\overline{u}_n(x,t)\in W(\Omega)$ satisfying $\|u-\overline{u}_n\|_{C(\Omega)}\le\varepsilon/\|\mathbb{L}\|$, where $\overline{u}_n(x,t)=\Sigma_{i=1}^{n}a_iH_i(x,t)$. From $\mathbb{L}$ is bounded, we get:

$$\|\mathbb{L}u-\mathbb{L}\overline{u}_n\|_{C(\Omega)}\le\|\mathbb{L}\|\|u-\overline{u}_n\|_{C(\Omega)}\le\varepsilon.\ \text{So, } \mathbb{L}u=F \text{ gives } \|F-\mathbb{L}\overline{u}_n\|_{C(\Omega)}\le\varepsilon$$

As a result:

$$\|F-\mathbb{L}u_n\|_{C(\Omega)}=\|F-\mathbb{L}\sum_{i=1}^{n}a_i^*H_i(x,t)\|_{C(\Omega)}\le\|F-\mathbb{L}\overline{u}_n\|_{C(\Omega)}\le\varepsilon$$

According to Definition 3, $u_n(x,t)$ is an ε-approximate solution of eq. (15).

Remark 3. To find the minimum value of:

$$\|\mathbb{L}\left[\sum_{i=1}^{n}a_iH_i(x,t)\right]-F(x,t)\|_{C(\Omega)} \tag{20}$$

we only need to compute $\mathbb{L}(H_i(x,t))$, that is, to compute $v_1=\mathbb{L}(Q_i(x)t)$, $v_2=\mathbb{L}(Q_i(x)t^2)$, $v_3=\mathbb{L}(Q_i(x)r_2(t,t_j))$, where v_1, v_2, v_3 can be obtained from the definition of $\mathbb{L}$.

Next, we change the problem of searching the minimum value of eq. (19) into solving a system of linear eqs. (23) which has one solution at least. And every solution of this system gives an ε-approximate solution of eq. (1).

To solve eq. (19), we will replace (19) by:

$$\min_{a_i\in\mathbb{R}}\left[\sum_{q=1}^{p}\sum_{i=1}^{n}a_i\mathbb{L}H_i(x_q,t_q)-F(x_q,t_q)\right]^2 \tag{21}$$

where $\{(x_q, t_q)\} \subset \Omega$. Putting:

$$\mathbf{g}_i = [\mathbb{L}H_i(x_1, t_1), \mathbb{L}H_i(x_2, t_2), \cdots, \mathbb{L}H_i(x_p, t_p)]^{\mathrm{T}} \in \mathbb{R}^{p+1}$$

$$\mathbf{F_1} = [F(x_1, t_1), F(x_2, t_2), \cdots, F(x_p, t_p)]^{\mathrm{T}} \in \mathbb{R}^{p+1}$$

where $i = 1, 2, \cdots, n$ and $\mathbb{R}^{p+1}$ is the Euclidean space of dimension $p+1$. Then (21) can be written:

$$\min_{a_i \in \mathbb{R}} \| \sum_{i=1}^{n} a_i \mathbf{g_i} - \mathbf{F_1} \|_{\mathbb{R}^{p+1}}^2 \tag{22}$$

Definition 4. Let $E = span\{\mathbf{g_1}, \mathbf{g_2}, \cdots, \mathbf{g_n}\}$, we call $\phi \in E$ the best approximation element of $\mathbf{F_1}$ in E, if:

$$\| \phi - \mathbf{F_1} \|_{\mathbb{R}^{p+1}} = \min_{S^* \in E} \| S^* - \mathbf{F_1} \|_{\mathbb{R}^{p+1}}$$

Similar to [17], we can prove the following two theorems.

Theorem 6. There is a unique vector $\phi \in E = span\{\mathbf{g_1}, \mathbf{g_2}, \cdots, \mathbf{g_n}\}$, such that:

$$\| \phi - \mathbf{F_1} \|_{\mathbb{R}^{p+1}} = \min_{S^* \in E} \| S^* - \mathbf{F_1} \|_{\mathbb{R}^{p+1}}$$

Remark 4. Theorem 6 shows that we can obtain at least an ε-approximate solution.

Theorem 7. Let $\mathbf{F_1} \in \mathbb{R}^{p+1}$, then $y \in E$ is the best approximation element of $\mathbf{F_1}$ in E and only if:

$$(\mathbf{F_1} - y, \mathbf{g_k})_{\mathbb{R}^{p+1}} = 0, \quad k = 1, 2, \cdots, n$$

Theorem 8. $y = \Sigma_{i=1}^{n} a_i^* \mathbf{g_i}$ is the best approximation element of (22) with respect to $\mathbf{F_1}$ in E and only if $\{a_i^*\}_{i=1}^{n}$ is the solution of the following equations which is the normal equation of (20):

$$\frac{\partial}{\partial a_k} \sum_{q=1}^{p} \left[\sum_{i=1}^{n} a_i^* g_i(x_q, t_q) - F(x_q, t_q) \right]^2 = 0, \quad k = 1, 2, \cdots, n \tag{23}$$

Proof: The system of eqs. (23) can be rewritten:

$$\sum_{i=1}^{n} a_i^* (\mathbf{g_i}, \mathbf{g_k})_{\mathbb{R}^{p+1}} = (\mathbf{F_1}, \mathbf{g_k})_{\mathbb{R}^{p+1}}, \quad k = 1, 2, \cdots, n$$

which is equivalent to $(\mathbf{F_1} - y, \mathbf{g_k})_{\mathbb{R}^{p+1}} = 0, k = 1, \cdots, n.$ So, the conclusion follows from Theorem 7.

Examples

In this section, the approach will be applied to the following example, which comes from [14]. We will compare numerical results with the reference to demonstrate the accuracy and efficiency of the proposed method. Denote by $u_n(x, t)$, $u(x, t)$ the approximate solution and exact solution of the following example, respectively.

We need the following maximum error estimation function:

$$E_{\max} = \max_{1\le j\le m}\left\{\left[\sum_{i=1}^{n}\left|u(x_i, t_j)-u_n(x_i, t_j)\right|^2\right]^{1/2}\right\} \tag{24}$$

where *n* and *m* denotes the numbers of the selected points on the spatial domain $0 \le x \le L$ and temporal domain $0 \le t \le T$, respectively.

Example 1. Considering the following equation:

$$\frac{\partial u(x, t)}{\partial t} = \mathrm{D}_t^{1-\gamma(x,t)}\frac{\partial^2 u(x, t)}{\partial x^2} + 2\mathrm{e}^x\left\{t - \frac{t^{1+\gamma(x,t)}}{\Gamma[2+\gamma(x, t)]}\right\}, \qquad 0 < t \leqslant 1, \qquad 0 < x < 1 \tag{25}$$

subject to the initial and boundary conditions:

$$u(x, 0) = 0, \qquad u(0, t) = t^2, \qquad u(1, t) = ct^2, \qquad 0 \le t \le 1 \qquad 0 \le x \le 1 \tag{26}$$

The exact solution of eq. (25) is $u(x, t) = e^x t^2$.

Tables 1 and 2 list the maximum errors of eq. (25) with (26), which are obtained by using the numerical schemes (2.14)-(2.17) and (7.7)-(7.10) of [14] and our new numerical method for various $\gamma(x, t)$ and $\tau = h^4$, $\tau^2 = h^4$, respectively. Where τ and *h* denote the time step and space step on the finite domain $0 \le x, t \le 1$, respectively, and $\gamma(x, t)$ satisfy $0 < \gamma(x, t) < 1$. From tabs. 1 and 2, it can be readily seen that our numerical results have a higher accuracy than [14]. So our method is superior to [14].

Table 1. Comparison of maximum errors $E_{\max}$ with [14] for $\tau = h^4 = 1/16$

$\gamma(x, t)$	$\tau = h^4 = 1/16$	
	Our method	[14]
$10 - (xt)^4/300$	7.12343E-5	4.3927E-4
$15 - (xt)^8/400$	7.83749E-5	4.9179E-4
$20 - (xt)^2/500$	7.34866E-5	5.1910E-4
$30 - (xt)^4/750$	7.79721E-5	5.2298E-4
$15 + \sin^8(xt)/400$	7.67822E-5	4.9179E-4
$\cos(xt) + (xt)/2/28$	7.00517E-5	5.2180E-4
$2^{(xt)^{1/2}} - \sin(xt)/50$	6.54628E-5	3.18114E-4
$22 - (xt)^2 + (xt^4)/550$	7.59869E-5	5.1986E-4
$10 - \cos^4(xt)/300$	8.02613E-5	4.0926E-4
$e^{(xt)^{1/2}} - 1.5\,(\sin)^{1/2}/50$	9.99547E-5	2.77111E-4

Table 2. Comparison of maximum errors E_{max} with [14] for $\tau^2 = h^4 = 1/16$ and $\tau^2 = h^4 = 1/18$

$\gamma(x, t)$	$\tau^2 = h^4 = 1/16$		$\tau^2 = h^4 = 1/18$	
	Our method	[14]	Our method	[14]
$10 - (xt)^4/300$	4.68079E-5	1.9032E-4	9.57326E-6	2.9272E-5
$15 - (xt)^8/400$	4.66218E-5	2.8569E-4	9.57901E-6	8.9489E-5
$20 - (xt)^2/500$	4.66477E-5	2.3457E-4	9.595E-6	2.8811E-5
$30 + (xt)^4/750$	4.65879E-5	3.1275E-4	9.5834E-6	9.66960E-5
$15 + \sin^8(xt)/400$	4.66428E-5	2.7903E-4	9.57868E-6	7.6216E-5
$\cos(xt) + (xt)/2/28$	4.69246E-5	2.8331E-4	9.58259E-6	9.2886E-5
$2^{(xt)^{1/2}} - \sin(xt)/50$	4.68221E-5	1.6004E-4	9.55438E-6	4.1236E-5
$22 - (xt)^2 + (xt^4)/550$	4.65981E-5	3.1368E-4	9.59067E-6	9.1263E-5
$10 - \cos^4(xt)/300$	4.67718E-5	3.3883E-4	9.53807E-6	1.7434E-5
$e^{(xt)^{1/2}} - 1.5\sin(xt)^{1/2}/50$	7.3110E-5	7.3110E-5	9.4674E-6	3.0528E-5

Nomenclature

$D_t^{1-\gamma(x,t)}$ – Caputo variable-order difference, [–]
L – real number, [–]
$\mathbb{L}$ – linear bound operator
m – integer, [–]
N, n – integer, [–]
$\mathbb{N}$ – integer set, [–]
p – real number, [–]
q – real number, [–]
$\mathbb{R}$ – real number set, [–]
T – real number, [–]
t – time, [s]
u – concentration, [molcm^{-3}]
x – displacement, [cm]
$\gamma(x, t)$ – variable order of fractional derivative, [–]

Greek symbol

Γ – gramma function, [–]

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