A NEW INTEGRAL TRANSFORM METHOD FOR SOLVING STEADY HEAT-TRANSFER PROBLEM

by

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In this paper, we propose a new integral transform method for the first time. It is used to find the solution for the differential equation in the steady heat-transfer problem. The proposed technology is accurate and efficient.

Key words: *heat transfer, integral transform, differential equation, analytical solution*

Introduction

The integral transforms have applications in various fields of science and engineer, such as fluid mechanics, viscoelasticity, chemistry, physics, and finance [1]. We recall the Laplace, Sumudu, and Elzaki transforms.

The Laplace transform of the function $\phi(\tau)$ is given [1]:

$$\Phi(s) = L[\phi(\tau)] = \int_0^\infty \phi(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau \tag{1}$$

provided the integral exists for some s, where L is the Laplace transform operator.

The Sumudu transform of the function $\phi(\tau)$ is given [2]:

$$\Phi(p) = S[\phi(\tau)] = \frac{1}{p}\int_0^\infty \phi(\tau)\mathrm{e}^{-\frac{\tau}{p}}\mathrm{d}\tau \tag{2}$$

provided the integral exists for some p, where S is the Sumudu transform operator.

The Elzaki transform of the function $\phi(\tau)$ is given [3]:

$$\Phi(\theta) = E[\phi(\tau)] = \theta\int_0^\infty \phi(\tau)\mathrm{e}^{-\frac{\tau}{\theta}}\mathrm{d}\tau \tag{3}$$

provided the integral exists for some θ, where E is the Elzaki transform operator.

The main aim of the paper is to present a new integral transform method for handling the differential equation in the steady heat-transfer problem.

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A new integral transform

In mathematics, any transform has the following form [4]:

$$\Phi(\theta) = \int_{\theta_1}^{\theta_2} \phi(\tau)\Pi(\tau, \theta)\mathrm{d}\tau \tag{4}$$

where $\Pi(\tau, \theta)$ is the kernel function, the input of the transform (4) is a function $\Pi(\tau, \theta)$ and the output of the transform (4) is another function $\Phi(\theta)$.

The inverse transform associated inverse kernel is given [4]:

$$\phi(\tau) = \int_{\tau_1}^{\tau_2} \Phi(\theta)\Pi^{-1}(\tau, \theta)\mathrm{d}\theta \tag{5}$$

Following idea of eq. (4), we consider the following kernel function $\Pi(\tau, \theta) = \mathrm{e}^{-\tau/\varpi}$.

Definition 1. The integral transform of the function $\phi(\tau)$ is defined:

$$\Phi(\varpi) = \mathrm{Y}[\phi(\tau)] = \int_0^\infty \phi(\tau)\mathrm{e}^{-\frac{\tau}{\varpi}}\mathrm{d}\tau\,,\ \tau > 0 \tag{6}$$

provided the integral exists for some ϖ, where $\theta \in (-\tau_1, \tau_2)$ and Y is the integral transform operator.

The properties of the integral transform are given:

(T1) Let $\Phi_1(\varpi) = \mathrm{Y}[\phi_1(\tau)]$ and $\Phi_2(\varpi) = \mathrm{Y}[\phi_2(\tau)]$

Then, we have:

$$\mathrm{Y}\left[a\phi_1(\tau) + b\phi_2(\tau)\right] = a\Phi_1(\varpi) + b\Phi_2(\varpi) \tag{7}$$

where a and b are two constants.

(T2) Suppose that $\Phi(\varpi)$=Y$[\phi(c\tau)]$, then we have:

$$\mathrm{Y}[\phi(c\tau)] = \frac{1}{c}\,\Phi\left(\frac{\varpi}{c}\right) \tag{8}$$

where c is a constant.

(T3) Let $\Phi(\varpi) = \mathrm{Y}[\phi(\tau)]$ and let the derivative of $\phi(\tau)$ be $\phi^{(1)}(\tau)$. Then, we have:

$$\mathrm{Y}[\phi^{(1)}(\tau)] = \frac{1}{\varpi}\,\Phi(\varpi) - \phi(0) \tag{9}$$

(T4) Let $\Phi(\varpi) = \mathrm{Y}[\phi(\tau)]$ and let the integral of $\phi(\tau)$ be $\int_0^\tau \phi(\tau)\mathrm{d}\tau$. Then, we have:

$$\mathrm{Y}\left[\int_0^\tau \phi(\tau)\mathrm{d}\tau\right] = \varpi\Phi(\varpi) \tag{10}$$

(T5)

$$\mathrm{Y}\left[\mathrm{e}^{c\tau}\right] = \frac{\varpi}{1 - c\varpi}, \tag{11}$$

where c is a constant.

(T6) $$\mathrm{Y}[1]=\varpi \tag{12}$$

Proof. (T1): With the help of the definition of the integral transform (6), we directly reduce to (T1).

(T2):

$$\mathrm{Y}[\phi(c\tau)]=\int_0^\infty \phi(c\tau)\mathrm{e}^{-\frac{\tau}{\varpi}}\mathrm{d}\tau=\frac{1}{c}\int_0^\infty \phi(c\tau)\mathrm{e}^{-\frac{c\tau}{c\varpi}}\mathrm{d}(c\tau)=\frac{1}{c}\Phi\left(\frac{\varpi}{c}\right)$$

(T3):

$$\mathrm{Y}[\phi^{(1)}(\tau)]=\left[\phi^{(1)}(\tau)\mathrm{e}^{-\frac{\tau}{\varpi}}\right]_0^\infty+\frac{1}{\varpi}\int_0^\infty \phi(\tau)\mathrm{e}^{-\frac{\tau}{\varpi}}\mathrm{d}\tau=\frac{1}{\varpi}\Phi(\varpi)-\phi(0)$$

(T4):

$$\mathrm{Y}\left[\int_0^\tau \phi(\tau)\mathrm{d}\tau\right]=\varpi\left[\left(\int_0^\tau \phi(\tau)\mathrm{d}\tau\right)\mathrm{e}^{-\frac{\tau}{\varpi}}\right]_0^\infty+\varpi\int_0^\infty \phi(\tau)\mathrm{e}^{-\frac{\tau}{\varpi}}\mathrm{d}\tau=\varpi\Phi(\varpi)$$

(T5):

$$\mathrm{Y}[\mathrm{e}^{c\tau}]=\int_0^\infty \mathrm{e}^{c\tau}\mathrm{e}^{-\frac{\tau}{\varpi}}\mathrm{d}\tau=\int_0^\infty \mathrm{e}^{-\left(\frac{1}{\varpi}-c\right)\tau}\mathrm{d}\tau=\frac{\varpi}{1-c\varpi}$$

(T6):

$$\mathrm{Y}[1]=\int_0^\infty \mathrm{e}^{-\frac{\tau}{\varpi}}\mathrm{d}\tau=\varpi$$

We finish the proof.

Solving the differential equation in the steady heat-transfer problem

We now consider the differential equation in steady heat-transfer problem [5]:

$$-hMT(x)=\rho Vc_pT^{(1)}(x) \tag{13}$$

subject to the initial condition:

$$T(0)=\gamma \tag{14}$$

where h is the convection heat transfer coefficient, M – the surface area of the body, ρ – the density of the body, V – the volume, c_p – the specific heat of the material, and $T(x)$ – the temperature.

Taking the integral transform of eq. (13) gives:

$$-hMT(\varpi) = \rho V c_p \left(\frac{1}{\varpi} T(\varpi) - T(0) \right) \tag{15}$$

which leads to the solution of eq. (13) in the form:

$$T(x) = \gamma \mathrm{e}^{-\frac{hM}{\rho V c_p} x} \tag{16}$$

Conclusion

In this work, we proposed a new integral transform, which is different from the Laplace-transform, Sumudu-transform, and Elzaki-transform operators. We used the technology to solve the differential equation in the steady heat-transfer problem. The proposed technology is, as alternative approach, accurate and efficient.

Nomenclature

c_p – specific heat of the material, [$\mathrm{Jkg^{-1}K^{-1}}$]
h – convection heat transfer coefficient, [$\mathrm{Wm^{-2}K^{-1}}$]
$T(x)$ – temperature, [K]
x – space co-ordinate, [m]

References

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