

Invited paper

ON THE ONSET OF INSTABILITIES IN A BENARD-MARANGONI PROBLEM IN AN ANNULAR DOMAIN WITH TEMPERATURE GRADIENT

by

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This manuscript addresses the linear stability analysis of a thermoconvective problem in an annular domain. The flow is heated from below, with a linear decreasing horizontal temperature profile from the inner to the outer wall. The top surface of the domain is open to the atmosphere and the two lateral walls are adiabatic. The effects of several parameters in the flow are evaluated. Three different values for the ratio of the momentum diffusivity and thermal diffusivity are considered: relatively low Prandtl number ($Pr = 1$), intermediate Prandtl number ($Pr = 5$) and high Prandtl number (ideally $Pr \rightarrow \infty$, namely $Pr = 50$). The thermal boundary condition on the top surface is changed by imposing different values of the Biot number, Bi . The influence of the aspect ratio (Γ) is assessed for through by studying several aspect ratios, Γ . The study has been performed for two values of the Bond number (namely $Bo = 5$ and 50), estimating the perturbation given by thermocapillarity effects on buoyancy effects. Different kinds of competing solutions appear on localized zones of the Γ - Bi plane. The boundaries of these zones are made up of co-dimension two points. Co-dimension two points are found to be function of Bond number, Marangoni number and boundary conditions but to be independent on the Prandtl number.

Key words: *Marangoni problem, thermocapillary convection, linear stability, buoyancy effects*

Introduction

It is well known that two different effects are responsible of the thermoconvective instabilities in fluid layers: gravity and capillarity forces. The problem in which both effects are considered, known as Benard-Marangoni (Bénard-Marangoni, BM) convection, has become a classical problem in fluid mechanics [1]. In the classical BM problem, heat is uniformly applied from the bottom wall and the solution becomes unstable for increasing temperature gradients. A more general problem includes the effect of horizontal temperature gradients resulting in new thermoconvective instabilities. Broadly speaking, the problem de-

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scribed could be treated as shallow water problem, see [2], but the objective of the current study is different since it relies on the analysis of the onset of instabilities.

The practical importance of the BM problem is nowadays widely recognized since it appears in a wide variety of processes such as the flow inside distillation columns, silicon crystal growth, film coating processes or the drying process by evaporation [3]. Mercier *et al.* [4] showed that the transition to a stationary convective mode can take place even if an adiabatic boundary condition is replaced with a Newton's law.

These instabilities have been analyzed considering either rectangular domains containing the flow [5, 6], either annular geometries [7], or infinite liquid films [8]. Literature includes some attempts to develop a theoretical framework for these problems, as presented in [9] and references therein, but for the moment it seems that more effort is required to develop more powerful numerical and mathematical tools to fully understand the process.

The following set of dimensionless numbers has been usually employed to characterize the different effects steering the behavior of the flow.

- Aspect ratio, $\Gamma = \delta/d$. It is the geometrical parameter that characterizes the domain.
- Prandtl number, Pr : the ratio of momentum diffusivity (kinematic viscosity) to thermal diffusivity. In this manuscript several Pr values are considered, ranging from unity to very high values (~ 50): $Pr = \nu/\kappa$.
- Marangoni number, Ma . It accounts for surface tension effects: $Ma = \gamma_T \Delta T d / \rho \kappa \nu$.
- Rayleigh number, Ra . It is representative of the importance of the buoyancy effects: $Ra = g \alpha \Delta T d^3 / \kappa \nu$.
- The two previously defined numbers are combined as the Bond number, Bo , which is the ratio of Rayleigh to Marangoni numbers, and thus represents the buoyancy against surface tension effects: $Bo = Ra/Ma = \rho g \alpha d^2 / \gamma_T$.
- Biot number, Bi , describes the heat transfer at the upper boundary condition between the fluid and the atmosphere. Values inside the range [0.2-3.2] have been considered in this work.

In the previous definitions, δ and d are the characteristic lengths of the domain that will be defined in the following section; γ_T stands for the rate of change of surface tension with temperature; ΔT is the temperature increment at the lower boundary with respect to the ambient temperature; ρ , κ , α , and ν are the density, the thermal diffusivity, the thermal expansion coefficient and the kinematic viscosity of the fluid, respectively, g is the acceleration due to gravity. It is convenient to remark that the Bond number used in the present work is the thermal Bond number defined as the ratio between thermogravitational effects and thermocapillarity effects, which should not be confused with the Bond number commonly used in interfacial systems. Since the objective of present study is to analyze phenomena in which the buoyancy is dominant, the Bond number range of interest includes values greater than one. The importance of heat-related parameters on the development of instabilities was analyzed in [10, 11]. More recently, the problem was also studied in annular geometries [12, 13] but neglecting the effect of heat transfer from the top surface and considering conduction through the lateral walls of the cylinder. Literature includes also works dealing with localized heating [14], or containers heated by a non-uniform flux [3]. Hoyas *et al.* [15] analyzed the effect of the Biot number on the different bifurcations for the case of $Pr = \infty$. The computational method was validated by comparing the results obtained with the experimental results by Garnier *et al.* [16]. The computational method was modified to deal with Prandtl numbers close to unity as shown in [17]. In [18], the authors investigated the existence of co-

dimension three bifurcations that are the points where co-dimensions two curves intersect on the Prandtl-Biot plane, and a new kind of instability was predicted. Those latter works dealt with the influence of the Biot number on the flow solutions. Also, the interest in understanding the influence of gravitational effects in thermo-convective phenomena has been rapidly growing [19-21]. Very recently the authors have studied the influence of the domain geometry on the onset of instabilities [22].

Depending on the symmetries of the growing perturbation, several bifurcations may appear. Up to six different competing solutions for the different wave numbers have been found, namely:

- stationary rolls (SR), similar to the ones of the basic state [23],
- hydrothermal wave of the first kind or oblique traveling waves (HWI) [23],
- longitudinal rolls (LR) [15],
- standing hydrothermal wave of second class or flower-like wave (HWII) [16, 24], and
- two new kinds of hydrothermal waves recently reported by Hoyas *et al.* [22] for low values of Γ .

The present work is devoted to analyze the effect of the gravitational and capillarity forces of the onset of flow instabilities by means of flow computations. It is worth to keep in mind that an adequate understanding of the way in which these instabilities are developed will create new possibilities of controlling them. In this work the effect of 3 different Prandtl numbers, ranging from a viscosity dominated problem ($Pr \approx \infty$, namely equal to 50) to problems in which the nonlinear flow effects are important ($Pr = 1$). An intermediate value of the Prandtl number representative of the conditions of water at ambient conditions ($Pr = 5$) is also considered. Additionally, the work is performed for two different Bond number conditions: $Bo = 50$ which means negligible thermocapillarity effects and $Bo = 5$ to evaluate the influence of the surface tension in perturbing the computed instabilities.

Model description and formulation

The physical domain considered in this work consists of a horizontal fluid layer of depth d (z coordinate) contained in the annular ring limited by two concentric cylinders of radii a and $a + \delta$ (r coordinate). A sketch of the domain is presented in fig. 1. The diameters of the two cylinders are chosen so that the bigger is the double of the smaller one, *i. e.*, $a = \delta$. The geometry of the fluid domain is characterized by the aspect ratio, Γ .

The bottom surface is considered to be rigid and is heated with a linearly decreasing (along the radius) temperature distribution with a temperature difference $T_G = 2$ K, which is kept constant throughout this study. The

reference temperature used in the definition of the Rayleigh and Marangoni numbers is the mean temperature difference between the bottom plate and the atmosphere, ΔT . The top sur-

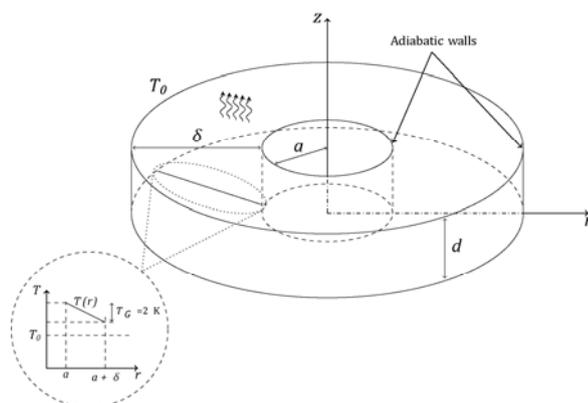


Figure 1. Sketch of the domain. Lateral walls are considered adiabatic; the fluid is heated from below and the top surface is open to the atmosphere

face is open to the atmosphere and the heat transfer to the atmosphere is expressed in terms of the Biot number (as shown in tab. 1). The two lateral walls of the cylinder are considered as adiabatic.

Table 1. Boundary conditions

$z = 0$	$z = d$	$r = a, a + \delta$
$u_r = 0$	$\partial_z u_r + \text{Ma} \partial_r \Theta = 0$	$u_r = 0$
$u_\phi = 0$	$r \partial_z u_\phi + \text{Ma} \partial_\phi \Theta = 0$	$u_\phi = 0$
$u_z = 0$	$u_z = 0$	$u_z = 0$
$\Theta = \Delta T - (T_G/\delta)r$	$\partial_z \Theta + \text{Bi} \cdot \Theta = 0$	$\partial_n \Theta = 0$

The fluid layer behavior can be described by means of the momentum and mass balance equations and the energy conservation principle. These equations are non-dimensionalized using d as characteristic length, d^2/κ as characteristic time and ΔT as characteristic temperature difference, as previously done in [10]. The equations become, respectively:

$$\nabla \cdot \mathbf{u} = 0 \quad (1)$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \text{Pr} \cdot (-\nabla p + \nabla^2 \mathbf{u} + \text{Ra} \Theta \mathbf{e}_z) \quad (2)$$

$$\partial_t \Theta + \mathbf{u} \cdot \nabla \Theta = \nabla^2 \Theta \quad (3)$$

In the equations governing the system \mathbf{u} is the velocity field with the three components expressed in cylindrical coordinates r, ϕ and z , i. e., u_r, u_ϕ and u_z . Θ is the flow temperature, and p is the pressure. In these equations the operators are expressed in cylindrical coordinates and \mathbf{e}_z is the unit vector in the z direction. The Boussinesq approximation is used for turbulence modeling as it is usual in this sort of problems [25]. Boundary conditions are similar to those of references [10, 17]. Non-slip wall condition (velocity equal to zero) is imposed on the lateral walls and on the bottom plate. A linearly varying temperature distribution is imposed on the bottom plate, while lateral walls are considered as adiabatic. On the top surface, the thermo-capillarity forces are modeled through the Marangoni condition [10], whereas the heat transfer to the atmosphere is simulated by the Biot condition. Notice that the dimensionless numbers presented in the previous chapter appear in eqs. (1-3) and in the boundary conditions summarized in tab. 1.

Numerical method

The temperature gradient on the bottom surface induces motion on the flow until a steady state, commonly known as basic state, is reached. Since the flow is laminar and due to the domain's symmetries, the basic state will be obtained under the assumption of 2D axisymmetric solution, and thus the dependency with ϕ can be neglected. Equations (1-3) developed for the cylindrical case then become:

$$r^{-1} \partial_r (r u_r) + \partial_z u_z = 0 \quad (4)$$

$$\text{Pr}^{-1} (u_r \partial_r u_r + u_z \partial_z u_r) = -\partial_r p + \Delta_c u_r - \frac{u_r}{r^2} \quad (5)$$

$$\text{Pr}^{-1} (u_r \partial_r u_z + u_z \partial_z u_r) = -\partial_z p + \Delta_c u_z + \text{Ra} \Theta \quad (6)$$

$$u_r \partial_r \Theta + u_z \partial_z \Theta = \Delta_c \Theta \quad (7)$$

where $\Delta_c = r^{-1} \partial_r (r \partial_r) + \partial_z^2$ is the Laplacian operator in cylindrical coordinates, simplified according to the previously mentioned symmetries.

The previous system of equations (4-7) can be solved in different ways, being spectral methods the most widely used [26]. Among the different spectral methods, the collocation method [27] is chosen in this work due to its accuracy and simplicity. The procedure is started by expanding the fluid variables in a truncated series of orho normal Chebyshev polynomials, as:

$$X^i(r, z) \approx \sum_{n=0}^N \sum_{m=0}^M a_{nm}^i \mathfrak{T}_n(r) \mathfrak{T}_m(z) \quad (8)$$

where $i = 1, \dots, 4$ and X^i stands for the four different flow variables, *i. e.*, p , u_r , u_z , and Θ , respectively. $\mathfrak{T}_j(x)$ is the Chebyshev polynomial of the first kind of degree j . The polynomial coefficients, a_{nm}^i , are now the unknowns of the problem with the superscript indicating the corresponding flow variables. The flow variables expanded expressions from eq. (8) are substituted into eqs. (1)-(3) and in the boundary conditions (tab. 1). The collocation method continues by evaluating the resultant equations in the Chebyshev-Gauss-Lobatto (CGL) points [28]. CGL points are defined as:

$$r_i = \cos\left(\pi \frac{i}{N}\right), \quad i = 0, 1, \dots, N$$

$$z_i = \cos\left(\pi \frac{i}{M}\right), \quad i = 0, 1, \dots, M$$

where N and M correspond to the order of the method in radial and axial direction, respectively. The use of CGL points is especially of interest when dealing with boundary effects [26] since the collocation points tend to concentrate near the boundaries. A specially developed procedure proposed by Mancho *et al.* [5], consisting in projecting the equations by the normal to the boundaries, is used to impose the boundary condition for the pressure. This procedure avoids the problem of the spurious modes as stated by Bernardi and Maday [29]. The pressure is determined with respect to an additive constant.

The non-linearity of the problem is solved by using a Newton-like iterative method, taking as first approach either the solution of the linearized problem, neglecting the nonlinear part of eqs. (5) and (6), or a previously known basic state “nearby” the new one. Convergence is typically obtained in less than 20 iterations as shown in the convergence test performed by Hoyas *et al.* [10].

The basic state is stable for low Ra. As Ra is increased (and thus the Ma for a given Bo), the basic state becomes unstable and several bifurcations arise. The purpose of this analysis is to determine the critical Ra and Ma values and the shape of growing instabilities for fixed Biot, Prandtl, and Bond numbers.

The flow stability is analyzed by perturbing the solution for the basic state with perturbation fields depending on the three cylindrical coordinates, r , φ , and z . Applying again the

axial symmetry of the problem, and thus periodicity along the azimuthal direction, φ , fluid magnitudes may be expanded in Fourier modes over φ as:

$$X(r, \varphi, z, t) = X_b(r, z) + X_p(r, z)e^{ik\varphi + \lambda t} \quad (9)$$

where subscripts b and p stand for the basic state and the induced perturbation, respectively; and $k \geq 0$ is the wave number. It is important to remark again that the basic state does not depend on φ .

The real part of the eigenvalue, λ , characterizes the stability. For $\text{Re}(\lambda) < 0$ the solution is stable since the perturbation tends to dissipate. For $\text{Re}(\lambda) > 0$ the solution is unstable. In this latter case, the bifurcation might be stationary (imaginary part of λ equal to zero) or oscillatory (when the imaginary part of λ is non-zero).

Once the Fourier modes of the flow variables, eq. (9), are substituted into the general eqs. (1)-(3) and the BCs from tab. 1, the problem is linearized as shown by [17]. The system then becomes:

$$[A]\bar{X} = \lambda[B]\bar{X} \quad (10)$$

Due to the boundary conditions the matrix $[B]$ is singular and thus not all the eigenvalues have a finite value. This issue is solved by using a transformation technique, developed by Navarro *et al.* [30], specifically designed for thermo-convective problems. The largest eigenvalue obtained through this transformation corresponds to the largest finite eigenvalue of the original problem. The computational code was validated experimentally in [15] and the generalization of the code to $\text{Pr} < 50$ was validated in [17], using the same procedure as in [10].

Results and discussion

The analysis intends to determine the critical Rayleigh number (Ra_c) and critical Marangoni number (Ma_c) for different conditions depending on the Biot, Prandtl, and Bond numbers and on the domain aspect ratio, Γ . The critical condition is found when the real part of the eigenvalue of eq. (10) is equal to 0 and is associated to a critical wave number, k . Solving the eigenvalue problem also allows obtaining the eigenvectors of the solution that represents the shape of the unstable modes.

Critical wave number

Figure 2 shows the critical wave number for each of the 6 combinations of Prandtl and Bond numbers analyzed in this work. The figure reports contour plots of the wave number for varying Γ and Biot number. Two different color scales are used: one for stationary modes going from light blue to yellow for increasing wave number and the other one for oscillatory modes going from light blue to dark blue for increasing wave number. Figure 2 shows the great variation of critical wave numbers and of the type of growing perturbation depending on the parameters object of the study. Moving along a constant Bi line for increasing Γ at low aspect ratios the modes are typically found to be steady then for a range of aspect ratios the modes are oscillatory and finally at higher aspect ratios the modes become again stationary. The wave number is found to increase with the aspect ratio suggesting that the characteristic wavelength of the disturbances is positively correlated with the domain depth. The precession of the modes along the domain for a limited range of Γ suggests that these aspect ratios should enable a better mixing and enhanced heat transfer performances.

Both increasing Pr and Bo , the Γ range which results in oscillatory solutions is reduced while the characteristic wavelength is slightly affected. The Prandtl number effect can be explained with the increase of damping due to viscous forces, while the Bo effect is related to the decrease of thermocapilarity forces at a given Ra . Increasing the Bi for all the other parameters fixed, results in a small decrease of the wave number and in the possible suppression of oscillations due to the decrease of the vertical temperature gradient through the domain.

Evolution of the critical Rayleigh number

Figure 3 reports the values of the critical Rayleigh number versus the domain aspect ratio Γ at Bi equal to 0.8 for three different cases. Going through fig. 3 from top to bottom, the cases are: $Pr = 1, Bo = 5$ corresponding to the case shown in fig. 2(e), $Pr = 1, Bo = 50$, corresponding to fig. 2(f), and $Pr = 50, Bo = 50$, corresponding to fig. 2(b). It is worth to remark that the value of the wave number along the curve is not continuous. Hollow circles stand for stationary bifurcations, whereas bold ones indicate oscillatory ones.

Additionally, the size of the marker is related with the value of the wave number, the smallest circle represents a value of $k = 0$ and the largest one a value of $k = 24$. The wave number increases with the aspect ratio of the domain, Γ (as also shown in fig. 1), while the critical Rayleigh number decreases. This same tendency was previously found by the authors in [22]. Figure 3 also includes representative top $r-\varphi$ plane isotherms plotted above the curves. The temperature field is non-dimensionalized with respect to the maximum Θ . Below the line the corresponding $z-r$ planes are also reported. Several types of instabilities are found along the constant Bi curves. As an example in fig. 3(b), for low aspect ratios, $\Gamma = 3 - 4$, stationary rolls (SR) are found, passing through a region of oscillatory Hydrothermal Wave of the first kind (HWI), for $\Gamma = 4 - 6.2$, and finally, for large aspect ratios, $\Gamma = 6.2 - 8$, longitudinal rolls (LR) appear as a result of the temperature gradient along the bottom wall which provides a stronger effect in the inner part of the domain.

Figure 3(a) reports in the same fashion as in fig. 3(b) the information for the case of $Pr = 1$ and $Bo = 5$, corresponding to fig. 2(c). For small aspect ratios a flower-like perturbation (as the ones obtained in [22]) appears instead of the stationary rolls of the former case. Following the curve, the arising perturbations are the same, passing through a region of oscillatory Hydrothermal Wave of the first kind (HWI), and finally for large aspect ratios, Longitudinal Rolls. It has to be remarked from comparison of figs. 3(a) and 3(b) that the critical wave number in the regions of HWI and LR is smaller in the former case. This means that the critical wave number is increased when the Marangoni number decreases, while the critical Rayleigh number follows the opposite tendency.

Comparing the curves from figs. 3(b) and 3(c) the effect of the Prandtl number can be assessed. The increase of the Prandtl number has a remarkable effect on the substantial increase of the critical Rayleigh number at which the modes become unstable due to the increase of viscous effects. The aspect ratio at which the different regimes appear is modified for $Pr > 1$; in fact the region of HWI is delayed (in aspect ratio) and becomes narrower, being the transition to LR also anticipated. Moreover, the viscous effects are found to be responsible of the arising of a flower-like perturbation for low aspect ratios.

Co-dimension two points

As figs. 2 and 3 illustrate, different kind of competing solutions appear on localized regions of the Γ - Bi plane. The boundaries of these zones are made up of co-dimension two points, where two of the competing solutions may appear at the same time. The evolution of

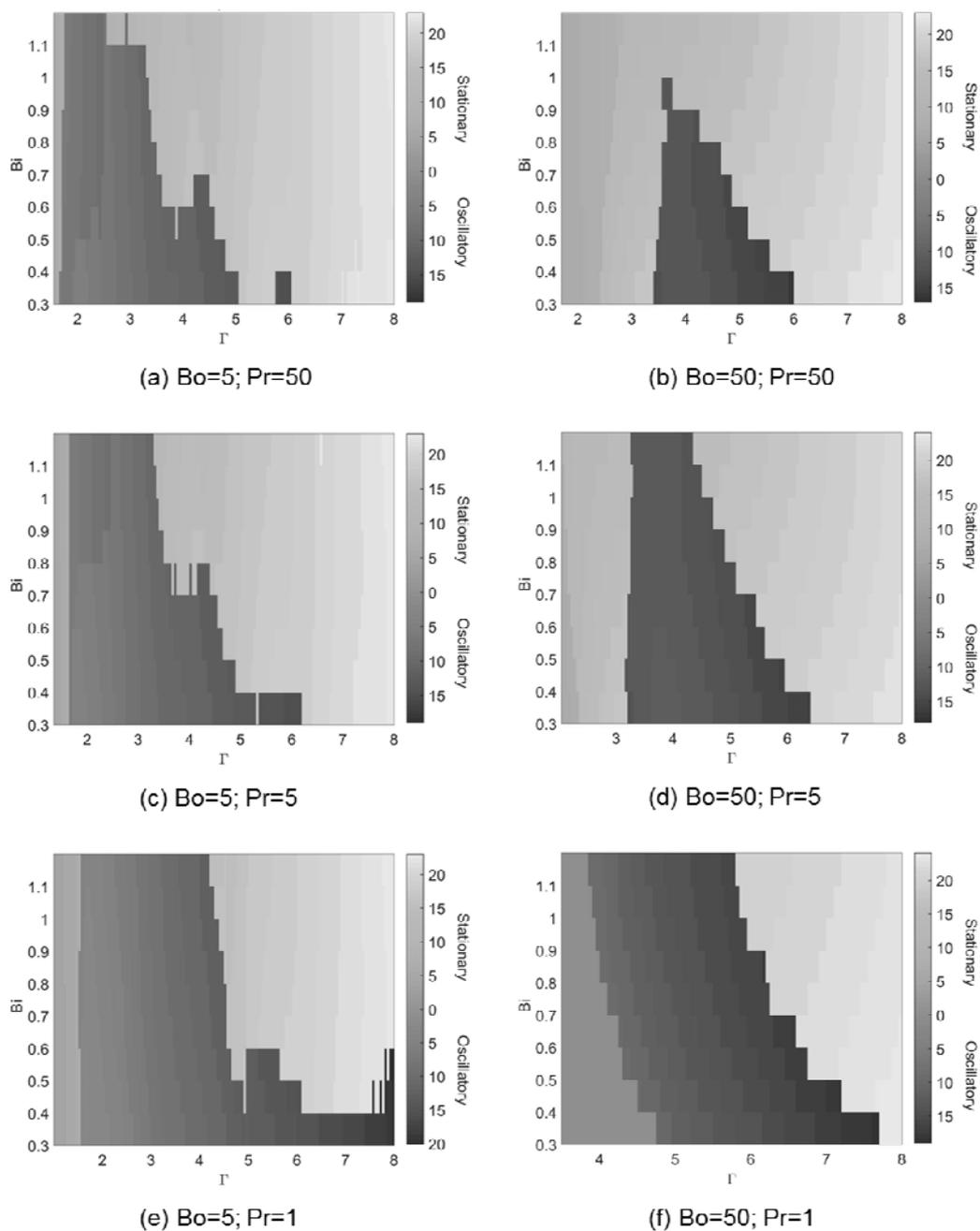


Figure 2. Critical wave number for the different numerical experiments analyzed in this work; the results are divided in stationary, $\text{Imag}(\lambda) = 0$, and oscillatory cases, $\text{Imag}(\lambda) \neq 0$; left column cases with $\text{Bo} = 5$ and for decreasing values of the Prandtl number and right column cases with $\text{Bo} = 50$ (for color image see journal website)

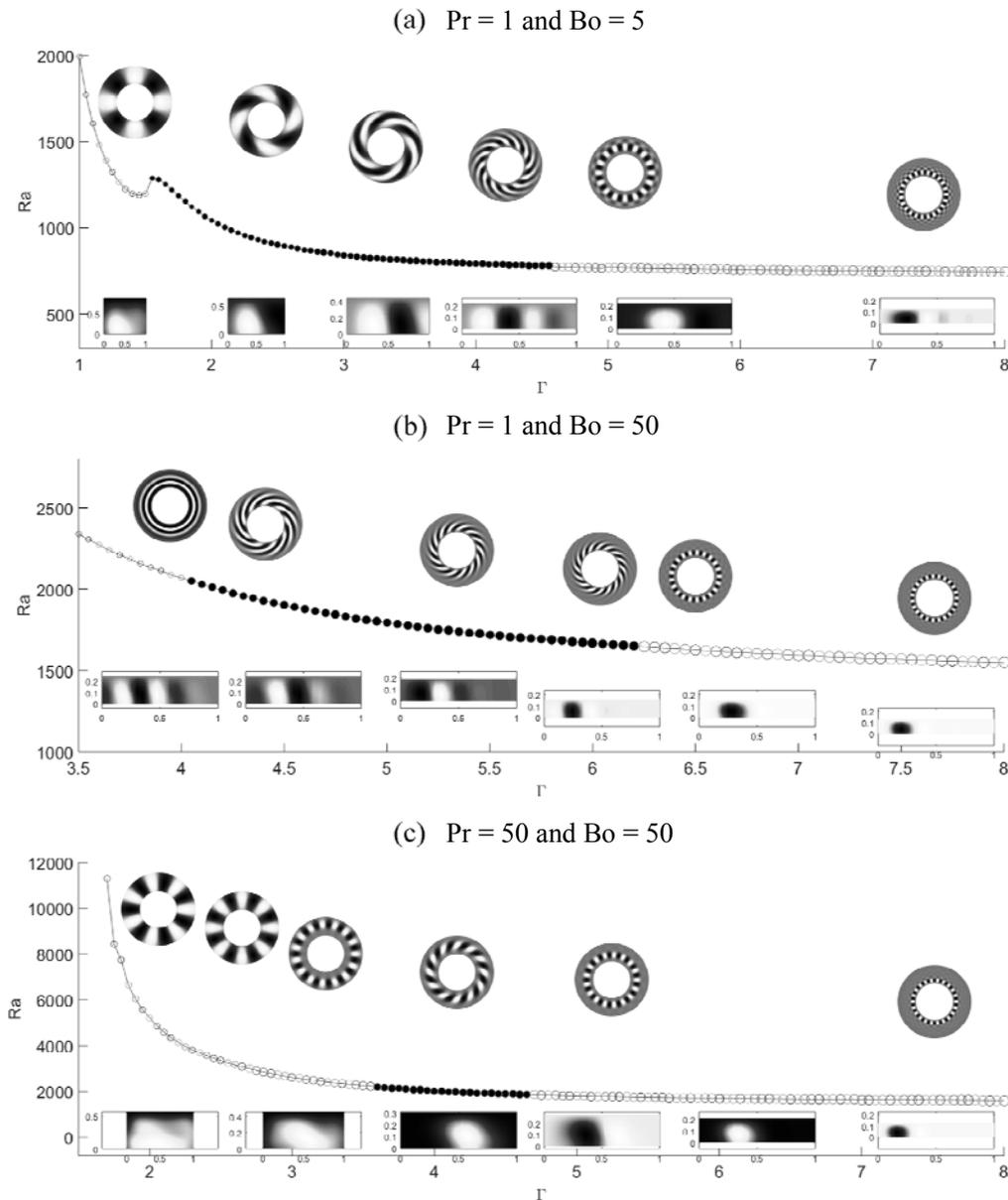


Figure 3. Evolution of the critical Ra with Γ for $Bi = 0.8$, and (a) $Pr = 1$ and $Bo = 5$, (b) $Pr = 1$ and $Bo = 50$, (c) $Pr = 50$ and $Bo = 50$. Empty circles stand for stationary bifurcations, whereas bold ones indicate oscillatory ones; the size of the circle indicates the wave number in a scale from $k = 0$ (the smallest circle) to $k = 24$ (the largest one) (for color image see journal website)

the critical Marangoni number and of the Biot number of all the co-dimension two points found in the present work is shown in fig. 4. Notice that all these points are obtained for different values of the aspect ratio Γ .

The main result shown in the figure is that most of the co-dimension two points have little dependence on the Prandtl number since the points seem to lay along two curves of equal Bond number: one for $Bo = 5$ and the other for $Bo = 50$. Of course at higher Bond numbers the critical Marangoni number is lower and practically constant since the instabilities are mostly triggered by the buoyancy effects and a slight increase of the critical Marangoni number is required to compensate the instabilizing effect of the decrease of the Biot number.

Some outliers with respect to the Ma - Bi curves are found among the low Bond cases for domains with high aspect ratio, where the Biot number is the dominant parameter as seen in previous works as [22]. On the other hand for high Bond numbers ($Bo = 50$), the flow is dominated by the thermoconvective phenomena, and the Biot number is not representative there.

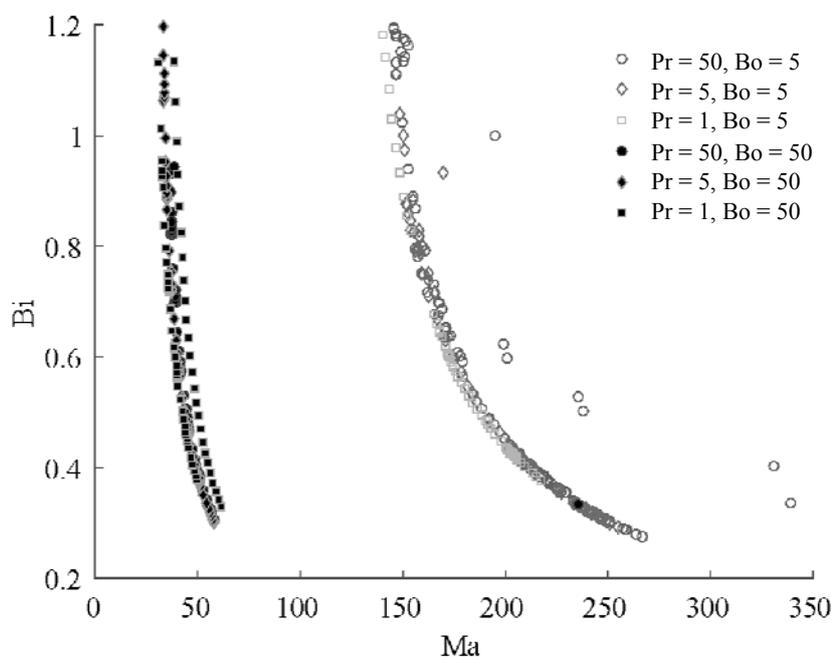


Figure 4. Representation of the critical Marangoni number and Biot number of all the co-dimension two points found in the present analysis (*for color image see journal website*)

Conclusions

This paper assesses the influence of the gravitational and capillarity forces of the onset of flow instabilities in a Bénard-Marangoni convection problem. Numerical simulations have been conducted in an annular domain with a temperature gradient imposed at its lower wall. This gradient induces a velocity field in the flow, that evolves until a basic state is reached. The analysis is performed for 3 different values of the Prandtl number, ranging from a viscosity dominated problem ($Pr = 50$) to problems in which the nonlinear flow effects are important ($Pr = 1$), and addressing an intermediate problem representative of water at ambient conditions ($Pr = 5$). This work also addresses the influence of the surface tension and of the thermocapillarity in perturbing the computed instabilities by analyzing two different values of the Bond number (5 and 50).

The critical wave number for each combination of Biot number on the free surface and domain aspect ratio has been studied. The perturbations are divided in stationary and oscillatory depending on the imaginary part of the eigenvalue. The increase of Γ allows to pass from a stationary unstable mode to an oscillatory one, then further increasing Γ to a stationary mode again. The wave number is found to be positively correlated with the domain aspect ratio.

It has been showed that different perturbation types share a common boundary in the Bi- Γ plane, these points where two competing solutions are found are known as co-dimension two points.

For a constant value of the Biot number (Bi = 0.8), the critical Rayleigh number generally decreases with the domain aspect ratio, while the critical wave number tends to increase. Different instabilities are found along the constant Bi line, mixing both oscillatory and stationary ones. The increase of Pr has a stabilizing effect on the flow which becomes unstable for higher Rayleigh numbers; as well the increase of Bo has an unstabilizing effect, resulting in higher wave numbers. The range of aspect ratios characterized by oscillatory solutions also increases with increasing Bo. For low Prandtl number, the types of the arising instabilities are different in the cases of Bo = 5 and Bo = 50, in particular for low aspect ratios, Γ between 3 and 4 where a higher Bond number promotes the appearance of Stationary Rolls.

The co-dimension two points present a tendency that does not depend on the Prandtl number since most of the points lay along two curves (one for each value of the Bond number considered in this work) in the Biot-Marangoni plane. It is shown that for the highest Bond number analyzed (Bo = 50), the thermocapillary effects are negligible and the Biot number plays a small role in the flow description. On the other hand for smaller Bond numbers (Bo = 5) some points lay out the curve. These cases are related with a high aspect ratio of the flow domain.

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