

## CONFORMABLE HEAT EQUATION ON A RADIAL SYMMETRIC PLATE

by

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*The conformable heat equation is defined in terms of a local and limit-based definition called conformable derivative which provides some basic properties of integer order derivative such that conventional fractional derivatives lose some of them due to their non-local structures. In this paper, we aim to find the fundamental solution of a conformable heat equation acting on a radial symmetric plate. Moreover, we give a comparison between the new conformable and the existing Grunwald-Letnikov solutions of heat equation. The computational results show that conformable formulation is quite successful to show the sub-behaviors of heat process. In addition, conformable solution can be obtained by an analytical method without the need of a numerical scheme and any restrictions on the problem formulation. This is surely a significant advantageous compared to the Grunwald-Letnikov solution.*

Key words: conformable derivative, conformable heat equation,  
Grunwald-Letnikov

### Introduction

In the non-classical theory of fractional heat conduction, the Fourier law is given as the time non-local dependence between the heat flux and the temperature gradient in the following constitutive equations with the *long-tail* power kernels:

$$\bar{q}(t) = -\frac{k}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{\alpha-1} \text{grad } T(\tau) d\tau, \quad 0 < \alpha \leq 1 \quad (1)$$

and

$$\bar{q}(t) = -\frac{k}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} \text{grad } T(\tau) d\tau, \quad 1 < \alpha \leq 2 \quad (2)$$

Equations (1) and (2) can be rewritten in terms of fractional integrals and derivatives as [1, 2]:

$$\bar{q}(t) = -k D_{RL}^{1-\alpha} \text{grad } T(t), \quad 0 < \alpha \leq 1 \quad (3)$$

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$$\vec{q}(t) = -kI^{\alpha-1} \text{grad } T(t), \quad 1 < \alpha \leq 2 \quad (4)$$

which yields to the following time-fractional heat conduction equation in terms of Caputo fractional derivative of order  $\alpha$ :

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\Delta T, \quad 0 < \alpha \leq 2 \quad (5)$$

The cases  $\alpha \rightarrow 0$ ,  $0 < \alpha < 1$ ,  $\alpha = 1$ ,  $1 < \alpha < 2$ , and  $\alpha = 2$ , respectively, correspond to localized heat conduction, sub (slow) heat conduction, standard heat conduction, super (fast) heat conduction and also ballistic heat conduction [3-5]. The Riemann-Liouville (RL) fractional integral and derivatives of order  $\alpha$  is defined as [6-8]:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0 \quad (6)$$

$$D_{\text{RL}}^\alpha f(t) = D^n I^{n-\alpha} f(t) = \frac{d^n}{dt^n} \left[ \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \right], \quad n-1 < \alpha \leq n \quad (7)$$

The Caputo fractional derivative is regularization of the RL fractional derivative in the time domain by incorporating the relevant initial conditions:

$$D_C^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, \quad n-1 < \alpha \leq n \quad (8)$$

Note that these are non-local operators expressed by convolution integrals with a weakly singular kernel. Due to this structure, a lot of implicit or explicit numerical schemes to analyze the fractional models in terms of non-local fractional operators have been developed because these operators are effective for modelling of sub/super processes in physical events [9-17].

Although the non-local fractional operators give the memory and hereditary effects that naturally occur in the real physical systems, the impossibility of analytical solutions or complexity of numerical approximations have led the researchers to introduce new local fractional order derivatives [18, 19]. Conformable fractional derivative is only one of these local operators proposed by Khalil *et al.* [20].

**(1) Definition:** Given a function  $f: [0, \infty) \rightarrow \mathbb{R}$ . Then the *conformable fractional derivative* of order  $\alpha$  is defined by:

$$(T_\alpha f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \quad (9)$$

for all  $t > 0$ .

Often,  $f^{(\alpha)}$  is used instead of  $T_\alpha(f)$  to denote the conformable derivative of  $f$  with fractional order  $\alpha$ . In addition, if the conformable derivative of  $f$  exists, then we simply say that  $f$  is  $\alpha$ -differentiable. If  $f$  is  $\alpha$ -differentiable in some  $(0, a)$  and  $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$  exists, then define:

$$f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$$

**(2) Theorem:** Let  $0 < \alpha \leq 1$ , and  $f$  and  $g$  be  $\alpha$  – differentiable at a point  $t > 0$ . Then:

$$(1) T_\alpha (af + bg) = aT_\alpha (f) + bT_\alpha (g) \quad \text{for all } a, b \in \mathbb{R}$$

$$(2) T_\alpha (t^p) = pt^{p-\alpha} \quad \text{for all } p \in \mathbb{R}$$

$$(3) T_\alpha (fg) = fT_\alpha (g) + gT_\alpha (f)$$

$$(4) T_\alpha \left( \frac{f}{g} \right) = \frac{gT_\alpha (f) - fT_\alpha (g)}{g^2}$$

$$(5) T_\alpha (\lambda) = 0, \text{ for all constant functions } f(t) = \lambda.$$

$$(6) \text{ If, in addition, } f \text{ is differentiable at a point } t > 0, \text{ then } T_\alpha (f)(t) = t^{1-\alpha} (df/dt).$$

In the recent years, there has been a growing interest to introduce the new properties and to reveal the new applications of conformable fractional derivative. Abdeljawad [21] proposed new notions such as sequential conformable derivative, fractional Laplace transform, integration by parts and fractional Taylor power series expansion. Abu Hammad and Khalil [22, 23] investigated the solutions of different types of conformable heat equations and introduced the fractional Fourier series with the applications. Khalil and Abu Shaab [24] also studied on some conformable differential equations. Atangana *et al.* [25] presented the conformable partial derivative and its properties and also the conformable Stokes and conformable Green's theorems in connection with the classical derivatives. Cenesiz and Kurt [26, 27] researched the solutions of conformable heat and wave equations. Avcı *et al.* [28] studied on a conformable fractional wave-like equation and so clarify some uncertain descriptions on Cauchy problems of these types of equations. The interested researchers are also referred to different applications of conformable operators [29-32].

In this study, we aim to find the fundamental solution of a Cauchy problem for a conformable heat equation defined on a radial symmetric plate. Furthermore, we are motivated to give a comparison between the conformable and the Grunwald-Letnikov (GL) solutions of a slow heat conduction process. Hence, we use the problem studied in [33] for this comparison and also show the retardation in the diffusion of heat by using the graphics.

### Formulation of the problem

Let us consider the following conformable heat equation defined on a radial symmetric plate:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(r, t) = \beta \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + f(r, t) \tag{10}$$

subjected to the following initial and boundary conditions given, respectively:

$$u(r, 0) = u_0(r), \quad 0 < r \leq R \tag{11}$$

and

$$u(0, t) = u(R, t), \quad t > 0 \tag{12}$$

where  $\partial^\alpha / \partial t^\alpha$  represents the conformable partial derivative with fractional order  $\alpha$  ( $0 < \alpha \leq 1$ ),  $\beta$  is thermal diffusivity coefficient, and  $f(r, t)$  denotes a heat source. We can find the fundamental solution as a combination of two separate Cauchy problems detailed in the following subsections.

*Homogeneous conformable heat equation and non-homogeneous initial condition*

Consider the following conformable partial differential equation:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(r, t) = \beta \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \quad (13)$$

with the initial and boundary conditions:

$$u(r, 0) = u_0(r), \quad 0 < r \leq R \quad (14)$$

$$u(0, t) = u(R, t) = 0, \quad t > 0 \quad (15)$$

We apply the separation of variables method and so assume that:

$$u(r, t) = Q(r)T(t) \quad (16)$$

By substituting eq. (16) into eq. (13) and after some mathematical computations, we obtain the following ordinary differential equations:

$$\frac{d^2 Q}{dr^2} + \frac{1}{r} \frac{dQ}{dr} + \mu^2 Q = 0 \quad (17)$$

$$\frac{d^\alpha T}{dt^\alpha} + \beta \mu^2 T = 0 \quad (18)$$

The bounded solutions of eq. (17) can be easily obtained:

$$Q_i(r) = c_1 J_0 \left( \frac{\lambda_i}{R} r \right) \quad (19)$$

where  $\mu_i = \lambda_i/R$  ( $i = 1, 2, \dots$ ) denotes the zeros of the first kind Bessel function  $J_0$  and  $c_1$  is arbitrary constant determined by boundary conditions. We use property (6) in Theorem and so the eq. (18) reduces to:

$$t^{1-\alpha} \frac{dT_i}{dt} + \beta \left( \frac{\lambda_i}{R} \right)^2 T_i = 0 \quad (20)$$

and its solution is:

$$T_i(t) = c_2 e^{-\beta \left( \frac{\lambda_i}{R} \right)^2 \frac{t^\alpha}{\alpha}} \quad (21)$$

Therefore, the fundamental solution of 1<sup>st</sup> problem is:

$$u(r, t) = \sum_{i=1}^{\infty} c J_0 \left( \frac{\lambda_i}{R} r \right) e^{-\beta \left( \frac{\lambda_i}{R} \right)^2 \frac{t^\alpha}{\alpha}} \quad (22)$$

where  $c = c_1 c_2$  is determined by the initial condition (14) and the orthogonality property of  $J_0$  Bessel function:

$$\int_0^R r J_0 \left( \frac{\lambda_i}{R} r \right) J_0 \left( \frac{\lambda_j}{R} r \right) dr = R^2 \begin{cases} 0, & i \neq j \\ \frac{J_1^2(\lambda_i)}{2}, & i = j \end{cases} \quad (23)$$

Consequently, the solution of 1<sup>st</sup> problem is obtained:

$$u(r, t) = \sum_{i=1}^{\infty} \frac{2}{R^2 J_1^2(\lambda_i)} J_0\left(\frac{\lambda_i}{R} r\right) e^{-\beta \left(\frac{\lambda_i}{R}\right)^2 \frac{t^\alpha}{\alpha}} \int_0^R r J_0\left(\frac{\lambda_i}{R} r\right) u_0(r) dr \quad (24)$$

Next, we consider the solution of 2<sup>nd</sup> problem as a similar manner.

*Non-homogeneous conformable heat equation and homogeneous initial condition*

Let us consider the following problem:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(r, t) = \beta \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + f(r, t) \quad (25)$$

with the initial and boundary conditions:

$$u(r, 0) = 0, \quad 0 < r \leq R \quad (26)$$

$$u(0, t) = u(R, t) = 0, \quad t > 0 \quad (27)$$

where  $f(r, t)$  is a heat source. Similar to 1<sup>st</sup> case,  $J_0 = [(\lambda_i/R)r]$ , ( $i = 1, 2, \dots$ ) are the basis functions of fundamental solution:

$$u(r, t) = \sum_{i=1}^{\infty} T_i(t) J_0\left(\frac{\lambda_i}{R} r\right) \quad (28)$$

We shall assume the heat source function is represented by:

$$f(r, t) = \sum_{i=1}^{\infty} f_i(t) J_0\left(\frac{\lambda_i}{R} r\right) \quad (29)$$

Note that, eq. (29) is not a particular assumption, *i. e.* we can achieve the same result by direct calculations. By substituting eqs. (28) and (29) into eq. (25), we have a Bessel differential equation as the same as eq. (17) and a conformable differential equation which is arranged by using the property (6) in Theorem (2) as:

$$t^{1-\alpha} \frac{dT_i}{dt} + \beta \left(\frac{\lambda_i}{R}\right)^2 T_i = f_i(t) \quad (30)$$

Equation (30) is a non-homogeneous ordinary differential equation with variable coefficients and its solution depends on the  $f_i(t)$ . Applying the orthogonality property of  $J_0$  and using eq. (29), we have that:

$$f_i(t) = \frac{2}{R^2 J_1^2(\lambda_i)} \int_0^R r J_0\left(\frac{\lambda_i}{R} r\right) f(r, t) dr \quad (31)$$

In our computations, we choose  $f(r, t) = A$  (constant) only for simplification. Hence, we get the solution of eq. (30):

$$T_i(t) = \frac{f_i(t) \left[ 1 - e^{-\beta \left(\frac{\lambda_i}{R}\right)^2 \frac{t^\alpha}{\alpha}} \right]}{\beta \left(\frac{\lambda_i}{R}\right)^2}, \quad (i=1, 2, \dots) \quad (32)$$

Finally, the fundamental solution of 2<sup>nd</sup> problem is in the following form:

$$u(r, t) = \sum_{i=1}^{\infty} \frac{2}{\beta (\lambda_i)^2 J_1^2(\lambda_i)} \left[ 1 - e^{-\beta \left(\frac{\lambda_i}{R}\right)^2 \frac{t^\alpha}{\alpha}} \right] J_0\left(\frac{\lambda_i}{R} r\right) \int_0^R r J_0\left(\frac{\lambda_i}{R} r\right) f(r, t) dr \quad (33)$$

The linear combination of 1<sup>st</sup> case solution (24) and the 2<sup>nd</sup> case solution (33) gives the whole solution of the main problem:

$$u(r, t) = \sum_{i=1}^{\infty} \frac{2}{J_1^2(\lambda_i)} J_0\left(\frac{\lambda_i}{R} r\right) \left\{ \frac{1}{R^2} e^{-\beta \left(\frac{\lambda_i}{R}\right)^2 \frac{t^\alpha}{\alpha}} \int_0^R r J_0\left(\frac{\lambda_i}{R} r\right) u_0(r) dr + \frac{1}{\beta (\lambda_i)^2} \left[ 1 - e^{-\beta \left(\frac{\lambda_i}{R}\right)^2 \frac{t^\alpha}{\alpha}} \right] \int_0^R r J_0\left(\frac{\lambda_i}{R} r\right) f(r, t) dr \right\} \quad (34)$$

Next, we validate the results graphically. The numerical results are held by introduction MATLAB codes.

### Numerical results

The physical behaviors of conformable heat solution are illustrated under the assumptions  $u_0(r) = \sin(\pi r)$ ,  $f(r, t) = 1$ , and  $R = 1$  for 2-D figures. Moreover, we fix the radius  $r = 0.5$  and obtain the results with respect to time,  $t$ , and also discretize the time interval into  $N = 100$  sub-intervals with the length of  $h = 0.01$ . In fig. 1, we first analyze the dependence of solution on the order of time conformable derivative by taking  $\alpha = 0.5, 0.75$ , and 1. This figure shows the retardation in the diffusion of heat by decreasing of fractional order  $\alpha$ . A significant analysis is given by fig. 2 in which a similarity between the conformable and the GL solutions appears as expected. We can clearly conclude that both of the conformable and GL models are so good generalizations to explain the slow diffusion of heat. As a comparison, we see that the settlement time of the solution changes with respect to the type of fractional order model.

Furthermore, we do not need any restrictions on conformable formulation. For instance, we arbitrarily choose the initial condition in the present paper and also do not apply a numerical approximation. Whereas, we had to use homogeneous initial condition to approximate the RL fractional derivative using GL numerical scheme in [33]. Therefore, we give the comparisons of the conformable and GL solutions by using the homogeneous initial condition and the heat source  $f(r, t) = 1$  in accordance with [33] for different values of  $\alpha$ . Figures (3a) and (3b) illustrate the surface solutions for  $\alpha = 0.75$  by choosing the initial conditions respectively, as  $u_0(r) = \sin(\pi r)$  and  $u_0(r) = r^2 - 1$ .

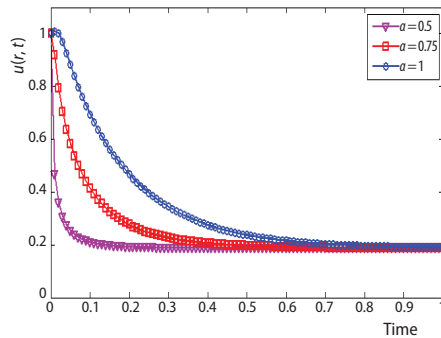


Figure 1. Dependence of conformable heat solution on the variation of  $\alpha$

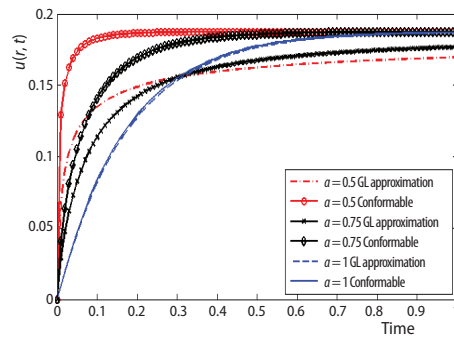


Figure 2. Comparison of conformable and GL solutions

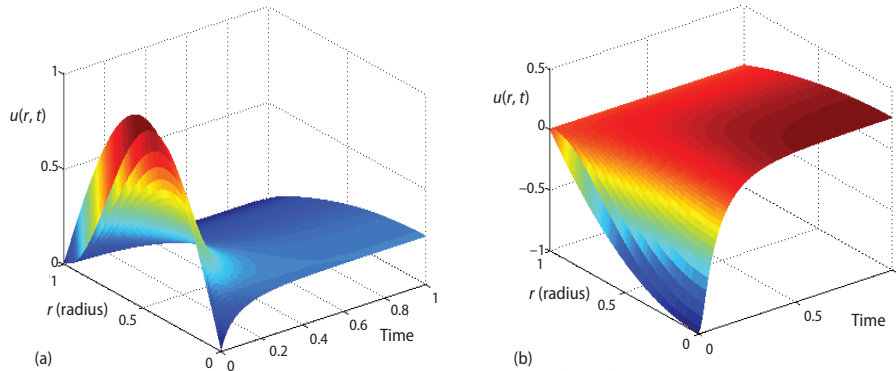


Figure 3. Surface plots for the initial conditions (3a)  $u_0(r) = \sin(\pi r)$  and (3b)  $u_0(r) = r^2 - 1$  (for color image see journal web site)

### Conclusion

In the recent years, local fractional operators have increasing interest among the researchers and have been successfully applied to the problems of mathematical physics and applied sciences. Conformable derivative is only one of these local operators which have been used to research the sub and super behaviours of heat and wave propagations. Moreover, this operator provides the basic properties of classical derivative such as product, quotient and chain rules, integration by parts, Rolle and mean value theorems, and so on. In this work, we consider a non-homogeneous conformable heat equation formulated on a radial symmetric plate and then use a direct computation technique to find the fundamental solution without the need of a numerical method. By the present study, we give attention to the comparison of local and non-local solutions by using MATLAB figures. Consequently, we should clarify that the conformable formulation is a good alternative to the well-known non-local fractional models of heat equation.

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