A NOTE ON CATTANEO-HRISTOV MODEL WITH NON-SINGULAR FADING MEMORY

by

Badr Saad T. ALKAHTANI[a*] and Abdon ATANGANA[b]

[a] Department of Mathematics, Colleges of Sciences, King Saud University, Riyadh, Saudi Arabia
[b] Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein, South Africa

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Using the new trend of fractional differentiation based on the concept of exponential decay law, the Cattaneo model of diffusion in elastic medium was extended by Hristov. This model displays more physical properties than the first version. However no solution of this new equation is suggested in the literature. Therefore, this paper is devoted to the analysis of numerical solution of the Cattaneo-Hristov model with non-singular fading memory.

Key words: *Cattaneo-Hristov model, non-singular fading memory, fractional derivative*

Introduction

Plates or media with elastic properties are important components that possess extremely significant mechanical structures and, as such, have a great collection of practical applications, more precisely in manufacturing building and microprocessor production [1-3]. For instance the reedy plates, which are rudimentary elements in numerous mechanical structures, can develop cranks because wear of the quantifiable or design. No wonder why many researchers have intensively reconnoitered the mechanical properties of such configurations in the last decades [4-8]. The model of transient heat was developed by Cattaneo [9, 10] and other reference [11]. Cattaneo, who generalized the well-known Fourier law via a linear superposition of the heat flux, suggested a dampening function connected to a restricted speediness of warmness dissemination in unbending conductors and its time derivative connected in its antiquity. With the use of Jeffrey kernel, the energy balance yields to the following Cattaneo equation [12]:

$$\frac{\partial T(x,t)}{\partial t} = \frac{b_2}{\gamma}\int_o^t \exp\left[-\left(\frac{t-k}{\gamma}\right)\right]\frac{\partial T(x,t)}{\partial x}\mathrm{d}k, \quad b_2 = \frac{k_2}{\rho c_p} \tag{1}$$

where, $0 < t < T$, and $a < x < b$. However, the mathematical equation describing the energy conservation of internal energy from Jeffrey type integro-differential equation:

$$\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial^2 T(x,t)}{\partial x^2} + \frac{a_2}{\gamma}\int_{-\infty}^t \exp\left(-\frac{t-k}{\gamma}\right)\frac{\partial^2 T(x,k)}{\partial x^2}\mathrm{d}k, \quad a_1 = \frac{k_1}{\rho c_p}, \quad a_2\,\frac{k_2}{\rho c_p} \tag{2}$$

* Corresponding author, e-mail: balgahtani1@ksu.edu.sa

Thus by considering the following change, we obtain $\chi = 1/\gamma$ the previous equation yields:

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + \chi a_2 \int_{-\infty}^{t} \exp\left[-\chi(t-k)\right] \frac{\partial^2 T(x,k)}{\partial x^2} \mathrm{d}k \tag{3}$$

Cattaneo-Hristov model

Recent, a step forward in fractional differentiation was suggested by Caputo and Fabrizio in their work [13], where they proposed a fractional derivative with a kernel similar to that of Jeffrey. This operator was found suitable in modeling some real world problems. The new derivative offers some new benefits that could not have been offered by classical fractional calculus based on Riemann-Liouville and Caputo derivatives. For instance, we do not have to worry about the singularity as the new kernel is based upon the law of exponential decay. The derivative helps us obtain the Laplace transform of some functions. In term, of power, the new derivative is a generalized power law, therefore, has more applicability than the power based on $x^{-\alpha}$. Since the introduction of these new derivatives, many researchers have been done. Atangana applied the new derivative to the model of non-linear Fisher's reaction-diffusion [14], Atangana and Nieto [15] suggested the numerical version of the new derivative using the Crank-Nicholson approach, Atangana and Alkahtani [16] applied the new derivative to the model of Keller-Segel, Gomez-Aguilar *et al.* [17] applied this derivative to model RC and RL electrical circuit models, Atangana and Iknur [18] model the dynamic of interaction between a bilingual person and a monolingual person. Goufo [19] applied this derivative to the Korteweg-de-Vries-Bergers model. In the same line of idea Hristov, using this derivative generated a model of diffusion in elastic medium with fading memory. The Cattaneo-Hristov model of diffusion is give as [4-6]:

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + a_2(1-\alpha)\,{}_0^{\mathrm{CF}}D_t^{\alpha}\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right] \tag{4}$$

We shall recall that the derivative of fractional order in Caputo-Fabrizio sense is give:

$${}_0^{\mathrm{CF}}\mathrm{D}_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t f'(y)\exp\left[-\frac{\alpha}{1-\alpha}(t-y)\right]\mathrm{d}y, \quad M(0)=M(1)=1, \quad \alpha\in[0,1) \tag{5}$$

Using the averaging method, Losada and Nieto [20] obtained M. It is important to note that, there are many ways to obtaining the function M.

In the literature, there is no sign of solution of this new eq. (4), therefore in this paper we present the numerical solution of the previous mentioned paper using the two numerical approaches.

Numerical solution

This section is dedicated to the derivation of the numerical solution of eq. (4) using different numerical approaches including the Crank-Nicholson scheme, implicit and explicit methods. The discritization is performed as follow, start with the discretization of the temporal domain [0, T] by putting a grid over the domain. The uniform grid is used with grid spacing $\Delta t = T/n$. The points within the grid are therefore given as t_k, $k = 0, 1, 2, 3, \ldots ,n$, with of course $t_k = k\Delta t$, $k = 0, 1, 2, 3, \ldots , n$. With the same idea with discritize the space with grid spacing:

$$\Delta x = \frac{X}{m}, \qquad x_j = j\Delta x, \qquad j = 0, 1, 2, 3, 4, \ldots, m$$

Implicit method applied to Cattaneo-Hristov model

We shall note that for implicit numerical approach the following approximations are obtained and will be used to provide the numerical solution of eq. (4):

$$\frac{\partial T}{\partial t}=\frac{T_i^j-T_i^{j-1}}{\Delta t}$$
$$\frac{\partial^2 T}{\partial x^2}=\frac{T_{i+1}^{j+1}-2T_i^{j+1}+T_{i-1}^{j+1}}{(\Delta x)^2} \tag{6}$$
$$\frac{\partial T}{\partial x}=\frac{T_i^j-T_{i-1}^j}{\Delta x}$$

Then the Caputo-Fabrizio approximation is given in the case of implicit scheme:

$${}_0^{\mathrm{CF}}\mathrm{D}_t^\alpha T=\frac{1}{\alpha}\sum_{k=0}^{j}\frac{T_i^k-T_i^{k-1}}{\Delta t}\delta_{i,j}$$
$$\delta_{i,j}=\exp\left[-\alpha\frac{\Delta t}{1-\alpha}(j-k)\right]-\exp\left[-\alpha\frac{\Delta t}{1-\alpha}(j-k+1)\right] \tag{7}$$

In this case the fading memory of the Cattaneo-Hristov model is approximated:

$$\mathrm{D}_t^\alpha\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right]=\frac{1}{1-\alpha}\int_0^t\frac{\partial}{\partial t}\left[\frac{\partial^2 T(x,y)}{\partial x^2}\right]\exp\left[-\frac{\alpha}{1-\alpha}(t-y)\right]\mathrm{d}y=$$
$$=\frac{1}{1-\alpha}\int_0^t\frac{\dfrac{\partial^2 T^j}{\partial x^2}-\dfrac{\partial^2 T^{j-1}}{\partial x^2}}{\Delta t}\exp\left[-\frac{\alpha}{1-\alpha}(t-y)\right]\mathrm{d}y=$$
$$=\frac{1}{1-\alpha}\int_0^t\frac{T_{i+1}^j-2T_i^j+T_{i-1}^j-T_{i+1}^{j-1}+2T_i^{j-1}-T_{i-1}^{j-1}}{\Delta t(\Delta x)^2}\exp\left[-\frac{\alpha}{1-\alpha}(t-y)\right]\mathrm{d}y=$$
$$=\frac{1}{1-\alpha}\sum_{k=0}^{j}\frac{T_{i+1}^k-2T_i^k+T_{i-1}^k-T_{i+1}^{k-1}+2T_i^{k-1}-T_{i-1}^{k-1}}{\Delta t(\Delta x)^2}\int_{t_k}^{t_{k+1}}\exp\left[-\frac{\alpha}{1-\alpha}(t_j-y)\right]\mathrm{d}y= \tag{8}$$
$$=\frac{1}{\alpha}\sum_{k=0}^{j}\frac{T_{i+1}^k-2T_i^k+T_{i-1}^k-T_{i+1}^{k-1}+2T_i^{k-1}-T_{i-1}^{k-1}}{\Delta t(\Delta x)^2}\varPhi_{i,j}$$
$$\varPhi_{i,j}=\exp\left[-\alpha\frac{\Delta t}{1-\alpha}(j-k)\right]-\exp\left[-\alpha\frac{\Delta t}{1-\alpha}(j-k+1)\right]$$

Replacing eq. (8), and (6) into eq. (4) we obtain:

$$\frac{T_i^j-T_i^{j-1}}{2\Delta t}=a_1\frac{T_{i+1}^{j+1}-2T_i^{j+1}+T_{i-1}^{j+1}}{(\Delta x)^2}+$$
$$+a_2(1-\alpha)\frac{1}{\alpha}\sum_{k=0}^{j}\frac{T_{i+1}^k-2T_i^k+T_{i-1}^k-T_{i+1}^{k-1}+2T_i^{k-1}-T_{i-1}^{k-1}}{2\Delta t(\Delta x)^2}\varPhi_{i,j} \tag{9}$$

For simplicity, we set:

$$c_1 = \frac{1}{2\Delta t}, \quad c_2 = \frac{a_1}{(\Delta x)^2}, \quad c_3 = \frac{a_2(1-\alpha)}{2\alpha\Delta t(\Delta x)^2}$$

to obtain

$$\begin{aligned}(2c_2 - c_1)T_i^{j+1} &= (-2c_2 - 2c_3\Phi_{i,j})T_i^j + c_2(T_{i+1}^{j+1} + T_{i-1}^{j+1}) + \\ &+ c_3\sum_{k=0}^{j-1}\left(T_{i+1}^k - 2T_i^k + T_{i-1}^k - T_{i+1}^{k-1} + 2T_i^{k-1} - T_{i-1}^{k-1}\right)\Phi_{i,j}\end{aligned} \tag{10}$$

Explicit method applied to Cattaneo-Hristov model

In this section, the explicit difference is used to solve numerically eq (4), to achieve this; we first present then numerical approximation with the explicit difference scheme.

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{T_i^{j+1} - T_i^j}{2\Delta t} \\ \frac{\partial^2 T}{\partial x^2} &= \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{(\Delta x)^2}\end{aligned} \tag{11}$$

Then the Caputo-Fabrizio approximation is given in the case of explicit scheme is given:

$$\begin{aligned}{}_0^{\text{CF}}\text{D}_t^\alpha T &= \frac{1}{\alpha}\sum_{k=0}^{j}\frac{T_i^{j+1} - T_i^j}{\Delta t}\delta_{i,j} \\ \delta_{i,j} &= \exp\left[-\alpha\frac{\Delta t}{1-\alpha}(j-k)\right] - \exp\left[-\alpha\frac{\Delta t}{1-\alpha}(j-k+1)\right]\end{aligned} \tag{12}$$

The fading memory of the Cattaneo-Hristov model is approximated for the explicit difference scheme is given:

$$\begin{aligned}\text{D}_t^\alpha\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right] &= \frac{1}{1-\alpha}\int_0^t \frac{\partial}{\partial t}\left[\frac{\partial^2 T(x,y)}{\partial x^2}\right]\exp\left[-\frac{\alpha}{1-\alpha}(t-y)\right]\text{d}y = \\ &= \frac{1}{1-\alpha}\int_0^t \frac{\dfrac{\partial^2 T^{j+1}}{\partial x^2} - \dfrac{\partial^2 T^j}{\partial x^2}}{\Delta t}\exp\left[-\frac{\alpha}{1-\alpha}(t-y)\right]\text{d}y = \\ &= \frac{1}{1-\alpha}\int_0^t \frac{T_{i+1}^{j+1} - 2T_i^{j+1} + T_{i-1}^{j+1} - T_{i+1}^j + 2T_i^j - T_{i-1}^j}{\Delta t(\Delta x)^2}\exp\left[-\frac{\alpha}{1-\alpha}(t-y)\right]\text{d}y = \\ &= \frac{1}{1-\alpha}\sum_{k=0}^{j}\frac{T_{i+1}^{k+1} - 2T_i^{k+1} + T_{i-1}^{k+1} - T_{i+1}^k + 2T_i^k - T_{i-1}^k}{\Delta t(\Delta x)^2}\int_{t_k}^{t_{k+1}}\exp\left[-\frac{\alpha}{1-\alpha}(t_j - y)\right]\text{d}y = \\ &= \frac{1}{\alpha}\sum_{k=0}^{j}\frac{T_{i+1}^{k+1} - 2T_i^{k+1} + T_{i-1}^{k+1} - T_{i+1}^k + 2T_i^k - T_{i-1}^k}{\Delta t(\Delta x)^2}\Phi_{i,j} \\ \Phi_{i,j} &= \exp\left[-\alpha\frac{\Delta t}{1-\alpha}(j-k)\right] - \exp\left[-\alpha\frac{\Delta t}{1-\alpha}(j-k+1)\right]\end{aligned} \tag{13}$$

Replacing eqs. (11), and (13) into eq. (4) we obtain:

$$\frac{T_i^{j+1}-T_i^j}{\Delta t}=a_1\frac{T_{i+1}^j-2T_i^j+T_{i-1}^j}{(\Delta x)^2}+ \\ +a_2(1-\alpha)\frac{1}{\alpha}\sum_{k=0}^{j}\frac{T_{i+1}^{k+1}-2T_i^{k+1}+T_{i-1}^{k+1}-T_{i+1}^k+2T_i^k-T_{i-1}^k}{\Delta t(\Delta x)^2}\Phi_{i,j} \tag{14}$$

For simplicity, we set:

$$c_1=\frac{1}{\Delta t},\quad c_2=\frac{a_1}{(\Delta x)^2},\quad c_3=\frac{a_2(1-\alpha)}{\alpha\Delta t(\Delta x)^2}$$

to obtain

$$(2c_1+\Phi_{i,j}c_3)T_i^{j+1}=(c_1-2c_2+c_3\Phi_{i,j})T_i^j+c_2(T_{i+1}^{j+1}+T_{i-1}^{j+1})+ \\ +c_3\sum_{k=0}^{j-1}\left(T_{i+1}^{k+1}-2T_i^{k+1}+T_{i-1}^{k+1}-T_{i+1}^k+2T_i^k-T_{i-1}^k\right)\Phi_{i,j} \tag{15}$$

The previous formula is the recursive formula used to generate the numerical representation, however in this paper it is not our main target.

Crank-Nicholson scheme applied to Cattaneo-Hristov model

Within the scope of Crank-Nicholson, the following approximations are given:

$$\frac{\partial T}{\partial t}=\frac{T_i^{j+1}-T_i^j}{2\Delta t} \\ \frac{\partial^2 T}{\partial x^2}=\frac{1}{2}\left[\frac{T_{i+1}^{j+1}-2T_i^{j+1}+T_{i-1}^{j+1}}{(\Delta x)^2}+\frac{T_{i+1}^j-2T_i^j+T_{i-1}^j}{(\Delta x)^2}\right] \tag{16}$$

the fading non-singular memory is approximate:

$$\mathrm{D}_t^\alpha\left[\frac{\partial^2 T(x,t)}{\partial x^2}\right]=\frac{1}{1-\alpha}\int_0^t\frac{\partial}{\partial t}\left[\frac{\partial^2 T(x,y)}{\partial x^2}\right]\exp\left[-\frac{\alpha}{1-\alpha}(t-y)\right]\mathrm{d}y= \\ =\frac{1}{1-\alpha}\int_0^t\frac{\dfrac{\partial^2 T^{j+1}}{\partial x^2}-\dfrac{\partial^2 T^j}{\partial x^2}}{\Delta t}\exp\left[-\frac{\alpha}{1-\alpha}(t-y)\right]\mathrm{d}y= \\ =\frac{1}{1-\alpha}\int_0^t\left[\begin{array}{c}\dfrac{T_{i+1}^{j+1}-2T_i^{j+1}+T_{i-1}^{j+1}-T_{i+1}^j+2T_i^j-T_{i-1}^j}{8\Delta t(\Delta x)^2}+ \\ +\dfrac{T_{i+1}^j-2T_i^j+T_{i-1}^j-T_{i+1}^{j-1}+2T_i^{j-1}-T_{i-1}^{j-1}}{8\Delta t(\Delta x)^2}\end{array}\right]\exp\left[-\frac{\alpha}{1-\alpha}(t-y)\right]\mathrm{d}y= \\ =\frac{1}{1-\alpha}\sum_{k=0}^{j}\left[\begin{array}{c}\dfrac{T_{i+1}^{k+1}-2T_i^{k+1}+T_{i-1}^{k+1}-T_{i+1}^k+2T_i^k-T_{i-1}^k}{8\Delta t(\Delta x)^2}+ \\ +\dfrac{T_{i+1}^k-2T_i^k+T_{i-1}^k-T_{i+1}^{k-1}+2T_i^{k-1}-T_{i-1}^{k-1}}{8\Delta t(\Delta x)^2}\end{array}\right]\int_{t_k}^{t_{k+1}}\exp\left[-\frac{\alpha}{1-\alpha}(t_j-y)\right]\mathrm{d}y= \tag{17}$$

$$=\frac{1}{\alpha}\sum_{k=0}^{j}\frac{T_{i+1}^{k+1}-2T_{i}^{k+1}+T_{i-1}^{k+1}+T_{i+1}^{k-1}-2T_{i}^{k-1}+T_{i-1}^{k-1}}{8\Delta t(\Delta x)^2}\Phi_{i,j}$$
$$\Phi_{i,j}=\exp\left[-\alpha\frac{\Delta t}{1-\alpha}(j-k)\right]-\exp\left[-\alpha\frac{\Delta t}{1-\alpha}(j-k+1)\right] \tag{17}$$

Therefore replacing eqs. (17), and (16) into (4) yields:

$$\frac{T_i^{j+1}-T_i^j}{2\Delta t}=\frac{1}{2}a_1\left[\frac{T_{i+1}^{j+1}-2T_i^{j+1}+T_{i-1}^{j+1}}{2(\Delta x)^2}+\frac{T_{i+1}^{j}-2T_i^{j}+T_{i-1}^{j}}{2(\Delta x)^2}\right]+$$
$$+a_2(1-\alpha)\frac{1}{\alpha}\sum_{k=0}^{j-1}\frac{T_{i+1}^{k+1}-2T_{i}^{k+1}+T_{i-1}^{k+1}+T_{i+1}^{k-1}-2T_{i}^{k-1}+T_{i-1}^{k-1}}{8\Delta t(\Delta x)^2} \tag{18}$$

With a parameters set before, we can reformulate the previous recursive formula:

$$(\mathrm{c}_1+2c_2+2c_3\Phi_{i,j})T_i^{j+1}=(c_1-2c_2)T_i^j+c_2(T_{i+1}^{j+1}+T_{i-1}^{j+1}+T_{i+1}^{j}+T_{i-1}^{j})+$$
$$+\sum_{k=0}^{j-1}\left(T_{i+1}^{k+1}+T_{i-1}^{k+1}+T_{i+1}^{k-1}-2T_i^{k-1}+T_{i-1}^{k-1}\right)\Phi_{i,j}c_3 \tag{19}$$

Conclusion

The new model of heat in elastic medium with fading memory that was recently developed by Hristov was considered in this work. The new model was solved numerically using three different numerical schemes including the explicit, implicit and Crank-Nicholson approaches.

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