ANALYTICAL MODELING OF THE THERMAL BEHAVIOR OF A THIN LUBRICANT FILM UNDER NON-LINEAR CONDITIONS

by

Najib LARAQI *, ***Cyril LAMRIBEN, Ali HOCINE, Zsolt Andrei PETER, and Abderezak HAMOUDA***

Laboratory of Thermal Interfaces and Environment (LTIE), Paris West University, Ville d'Avray, France

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Lubrication is an important phenomenon in a wide field of industry such as automotive, aerospace, mechanical transmission systems, and many others. The viscosity of fluid is a determining factor in the thermal behaviour of lubricant and solid surfaces in friction. In practice the viscosity varies strongly as a function of local pressure and temperature. In this study we are interested in the effect of temperature on the viscosity and the thermal behavior of the lubricant. We solve the dynamic and energy equations under non-linear conditions considering that the viscosity decreases following an exponential law of the temperature as it is known in the literature, $\mu = \mu_0 \, \mathrm{e}^{-\beta(T-T_0)}$.
The analytical solution is compared to a numerical modelling using a finite difference methods. The results show an excellent agreement.
We analyse the effect of the viscosity coefficient, β, on the velocity and the temperature in the thin lubricant film.

Key words: *non-linear lubrication, heat transfer in a thin lubricant film, analytical modelling*

Introduction

Temperature plays an important role in the lubrication phenomenon. The thermal behaviour of lubricant, particularly for thin films, controls the performance of frictional devices. This problem is encountered in several industrial systems such as gear, bearings, journal bearing, metal forming, and others. The heat flux generated by friction, and also the velocity and temperature of the film depend strongly of the viscosity of the lubricant. The latter varies strongly as a function of local pressure and temperature. Several studies were developed in the literature to analyse the thermo-hydrodynamic phenomenon of lubricants [1-4]. All show that the nature of lubricant and its temperature dependence are determining in the behaviour of frictional systems. Several studies show that friction causes very high flash temperature (*e. g.*, [5-7]).

An analytical study was developed recently to analyse the effect of the velocity of wall on the temperature of a thin lubricant film [8]. The viscosity was considered independent of the temperature. In practice the viscosity varies strongly as a function of the local temperature. This

* Corresponding author, e-mail: nlaraqi@u-paris10.fr

behaviour leads to solving non-linear equations. Several studies were devoted to a non-linear phenomenon in heat transfer [9-11]. Authors suggested various methods to solve the non-linear governing equations. Each method depends of the nature of the non-linearity.

In this study we are interested in the effect of temperature on the viscosity and the thermal behaviour of the lubricant. We solve the dynamic and energy equations under non-linear conditions considering that the viscosity decreases following an exponential law of temperature as it is known in the literature.

The analytical solution is compared to a numerical modelling using a finite difference methods (FDM). The results show an excellent agreement.

We analyse the effect of the viscosity coefficient, β, on the velocity and the temperature in the thin lubricant film.

Problem description

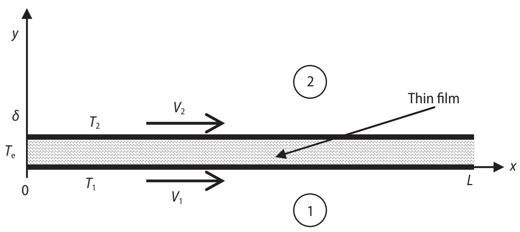

Figure 1. Simplified schema of a lubricant film between two moving solids

Consider the frictional system which consists of two moving bodies (1) and (2), fig. 1, separated by a thin lubricant film, with depth, δ, (the order of magnitude of a micron) and the length, L, ($L \gg \delta$). Heat flux is generated by shear of the lubricant. We consider that the temperature at the entrance (x = 0) is uniform and equal to T_e, and the walls (1) and (2) are at a uniform temperature T_1 and T_2, respectively.

Governing equations

The behaviour of the lubricant film is governed by the momentum eq. (1) which is simplified and the energy eq. (2). The dynamic viscosity is dependent of the local temperature following an exponential law eq. (3):

$$\frac{\partial}{\partial y}\left[\mu(T)\frac{\partial u}{\partial y}\right] = \frac{\partial P}{\partial x} \tag{1}$$

$$\rho c u(y)\frac{\partial T}{\partial x} = \lambda \frac{\partial^2 T}{\partial y^2} + \mu(T)\left(\frac{\partial u}{\partial y}\right)^2 \tag{2}$$

$$\mu = \mu_0 \mathrm{e}^{-\beta(T-T_0)} \tag{3}$$

The boundary conditions are given by the following equations:

$$u(0) = V_1, \quad u(\delta) = V_2 \tag{4}$$

$$T(x,0) = T_1, \quad T(x,\delta) = T_2, \quad T(0,y) = T_\mathrm{e} \tag{5}$$

As a first step, we consider that $\partial P/\partial x = 0$, so eq. (1) becomes:

$$\frac{\partial}{\partial y}\left[\mu(T)\frac{\partial u}{\partial y}\right] = 0 \tag{1a}$$

Usually, the convection term is less than the conduction one, then we can consider that:

$$\rho c u(y)\frac{\partial T}{\partial x} \ll \lambda \frac{\partial^2 T}{\partial y^2}$$

in eq. (2).

Equation (2) becomes:

$$\lambda \frac{\partial^2 T}{\partial y^2} = -\mu(T)\left(\frac{\partial u}{\partial y}\right)^2 \tag{2a}$$

Analytical solution

By integrating eq. (1a) it follows that:

$$\frac{\partial u}{\partial y} = \frac{A}{\mu_0}\mathrm{e}^{\beta(T-T_0)} \tag{1b}$$

where A is a constant.

Substituting eq. (1b) in eq. (2a) we obtain:

$$\lambda \frac{\partial^2 T}{\partial y^2} = -\frac{A^2}{\mu_0}\mathrm{e}^{\beta(T-T_0)} \tag{2b}$$

Multiplying eq. (2b) by $\partial T/\partial y$, and integrating leads to:

$$T(y) - T_0 = \frac{1}{\beta}\ln\left[\frac{\lambda\mu_0 C}{2A^2}\left\{\frac{1}{ch^2\left[\frac{\sqrt{\beta C}}{2}(y+D)\right]}\right\}\right] \tag{2c}$$

where C and D are constants.

Substitution eq. (2c) in eq. (1b) we find the velocity profile:

$$\frac{\partial u}{\partial y} = \frac{\lambda C}{2Ach^2\left[\frac{\sqrt{\beta C}}{2}(y+D)\right]}$$
$$u(y) = \frac{\lambda\sqrt{C}}{A\sqrt{\beta}}th\left[\frac{\sqrt{\beta C}}{2}(y+D)\right] + B \tag{1c}$$

where B is a constant.

The constants A, B, C, and D are determined by using the boundary conditions given by eqs. (4) and (5).

Particular case: $T_1 = T_2 = T_0$

Considering that $T_1 = T_2 = T_0$, we obtain all the constants under an explicit form:

$$A=\frac{8\lambda}{\beta\delta\left(V_2-V_1\right)}\frac{th^{-1}\left\{\left[\dfrac{8\lambda}{\beta\mu_0\left(V_2-V_1\right)^2}+1\right]^{-1/2}\right\}}{\left[\dfrac{8\lambda}{\beta\mu_0\left(V_2-V_1\right)^2}+1\right]^{1/2}},\quad B=\frac{V_1+V_2}{2}$$

$$C=\frac{16}{\beta\delta^2}\left[th^{-1}\left\{\frac{1}{\sqrt{\dfrac{8\lambda}{\beta\mu_0\left(V_2-V_1\right)^2}+1}}\right\}\right]^2,\quad D=-\frac{\delta}{2}\tag{6}$$

and the solutions (1c) and (2c) become:

$$u(y)=\frac{V_2-V_1}{2}\sqrt{\frac{8}{\beta^*}+1}\,th\left[\left(2\frac{y}{\delta}-1\right)th^{-1}\left(1/\sqrt{\frac{8}{\beta^*}+1}\right)\right]+\frac{V_2+V_1}{2}\tag{1d}$$

$$T(y)-T_0=\frac{1}{\beta}\ln\left\{\frac{\dfrac{\beta^*}{8}+1}{ch^2\left[\left(2\dfrac{y}{\delta}-1\right)th^{-1}\left(1/\sqrt{\dfrac{8}{\beta^*}+1}\right)\right]}\right\}\tag{2d}$$

Considering the following dimensionless quantities:

$$y^*=\frac{y}{\delta},\quad u^*=\frac{u-V_1}{V_2-V_1},\quad T^*=\frac{T(y)-T_0}{\Delta T_r},\quad \Delta T_r=\frac{\mu_0\left(V_2-V_1\right)^2}{\lambda},\quad \beta^*=\beta\frac{\mu_0\left(V_2-V_1\right)^2}{\lambda}\tag{7}$$

eqs. (1d) and (2d) become:

$$u^*=\frac{1}{2}\sqrt{\frac{8}{\beta^*}+1}\,th\left[\left(2y^*-1\right)th^{-1}\left(1/\sqrt{\frac{8}{\beta^*}+1}\right)\right]+\frac{1}{2}\tag{1e}$$

$$T^*=\frac{1}{\beta^*}\ln\left\{\frac{\dfrac{\beta^*}{8}+1}{ch^2\left[\left(2y^*-1\right)th^{-1}\left(1/\sqrt{\dfrac{8}{\beta^*}+1}\right)\right]}\right\}\tag{2e}$$

Numerical modeling for validation

To validate the analytical solution, the governing eqs. (1) and (2) are solved by using a numerical modelling based on the FDM. The mesh is denoted *i* in the x-direction and *j* in the y-direction. The space steps are Δx and Δy, respectively.

After the discretization of the governing equations we use an iterative method for their solving.

First, consider the momentum eq. (1) which can also be written:

$$\mu(T)\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\frac{\mathrm{d}\mu}{\mathrm{d}T}\frac{\partial T}{\partial y}=\frac{\partial P}{\partial x} \tag{8}$$

The discretization of eq. (8) by the FDM leads to:

$$\mu_{i,j}\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{\Delta y^2}+\left(\frac{\mathrm{d}\mu}{\mathrm{d}T}\right)_{i,j}\frac{u_{i,j}-u_{i,j-1}}{\Delta y}\frac{T_{i,j}-T_{i,j-1}}{\Delta y}=\frac{\partial P}{\partial x}=C \tag{9}$$

The viscosity $\mu_{i,j}$ and its derivative $(\mathrm{d}\mu/\mathrm{d}T)_{i,j}$ are explicit functions of the temperature from eq. (3).

To prepare the iterative process, we extract $u_{i,j}$ from eq. (9) such as:

$$u_{i,j}=\frac{u_{i,j+1}+u_{i,j-1}\left[1-\frac{1}{\mu_{i,j}}\left(\frac{\mathrm{d}\mu}{\mathrm{d}T}\right)_{i,j}\left(T_{i,j}-T_{i,j-1}\right)\right]-\frac{\partial P}{\partial x}\frac{\Delta y^2}{\mu_{i,j}}}{2-\frac{1}{\mu_{i,j}}\left(\frac{\mathrm{d}\mu}{\mathrm{d}T}\right)_{i,j}\left(T_{i,j}-T_{i,j-1}\right)} \tag{10}$$

The discretization of energy eq. (2) leads to:

$$\rho c u_{i,j}\frac{T_{i,j}-T_{i-1,j}}{\Delta x}=\lambda\frac{T_{i,j-1}-2T_{i,j}+T_{i,j+1}}{\Delta y^2}+\mu_{i,j}\left[\frac{u_{i,j+1}-u_{i,j-1}}{2\Delta y}\right]^2 \tag{11}$$

That gives $T_{i,j}$ under the following form:

$$T_{i,j}=\frac{T_{i,j-1}+T_{i,j+1}+T_{i-1,j}\left(\frac{\rho c u_{i,j}\Delta y^2}{\lambda\Delta x}\right)+\frac{\mu_{i,j}}{4\lambda}\left(u_{i,j+1}-u_{i,j-1}\right)^2}{\left(2+\frac{\rho c u_{i,j}\Delta y^2}{\lambda\Delta x}\right)} \tag{12}$$

Equations (10) and (11) are completed by the boundary conditions given by eqs. (4) and (5).

To solve the systems (10) and (11) we use the iterative method (successive over relaxation - SOR) which is a generalization of and improvement on the Gauss-Seidel method. The principal of this method consists in the computing a given function at the iteration $(k+1)$ $f_{i,j}^{(k+1)}$ by using a linear combination between the value at the previous iteration (k) $f_{i,j}^{(k)}$ and that estimated during the iterative process $f_{i,j}$. The latter uses a combination of temperatures computed at the iterations (k) and $(k+1)$. A relaxation coefficient ω is introduced for this process such as:

$$\begin{aligned}u_{i,j}^{(k+1)}&=(1-\omega)u_{i,j}^{(k)}+\omega u_{i,j}\\T_{i,j}^{(k+1)}&=(1-\omega)T_{i,j}^{(k)}+\omega T_{i,j}\end{aligned} \tag{13}$$

Results

Validation

To validate the proposed analytical solution we consider a general case for which the temperature of walls are different and the viscosity varies strongly as a function of the temperature. We use the data given by tab. 1.

For this case, we determine the values of the constants A, B, C, and D such as: $A = 4.473389025 \cdot 10^5$, $B = 37.97869477$, $C = 1.7105542822 \cdot 10^{14}$, and $D = -8.7003788006 \cdot 10^{-7}$.

Table 1. Data used for the comparison between analytical and numerical solutions

V_1 [ms^{-1}]	V_2 [ms^{-1}]	δ [m]	μ_0 [kgm^{-1}s^{-1}]	λ [Wm^{-1}K^{-1}]	β [K^{-1}]	T_0 [ºC]	T_1 [ºC]	T_2 [ºC]
0	50	10^{-6}	0.085	0.5	0.14	80	80	100

Figures 2 and 3 compare the analytical and numerical results for velocity and temperature profiles, respectively. This comparison shows an excellent agreement between both methods.

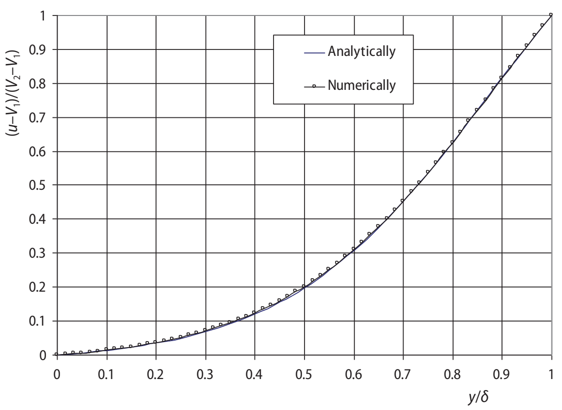

Figure 2. Comparison of analytical and numerical solutions for the velocity profile

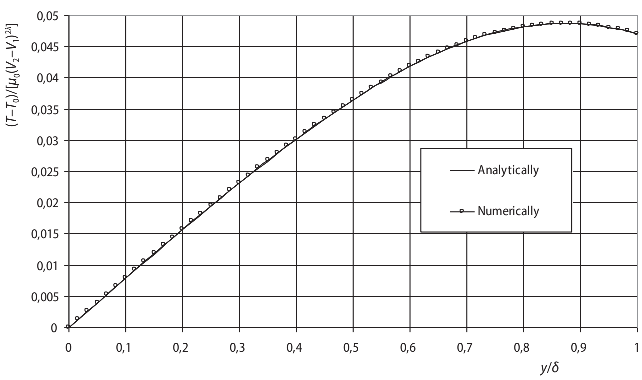

Figure 3. Comparison of analytical and numerical solutions for the temperature profile

Particular case: $T_1 = T_2 = T_0$

The velocity profile through the thickness of the film is given by fig. 4 for different values of the dimensionless viscosity coefficient β^*. The particular value $\beta^* = 0$ corresponds to the case of a constant viscosity $\mu(T) = \mu_0 = C$ for which the velocity profile is linear (Couette flow). More β^* value increases more the velocity profile is deformed and approaches the walls velocities.

Temperature profiles for the same conditions that those of the velocity are given by fig. 5. Because $T_1 = T_2$ in this particular case, the maximum of the temperature is located at $y = \delta/2$. The increasing in β^* leads to the decreasing in the viscosity and then in the temperature level.

Other cases with: $T_1 \neq T_2$

For these cases, we use the data given in tab. 1 and we consider several values of the viscosity coefficient, β. Considering the boundary conditions given by eqs. (4) and (5) we determine the values of constants A, B, C, and D. Table 2 gives the values of these constants for each value of β.

Table 2. Values of *A*, *B*, *C*, and *D vs.* *β* values

β [K^{-1}]	A	B	C	D
0.07	993690.8567	30.134790	$2.975381267 \cdot 10^{14}$	$-6.92414432 \cdot 10^{-7}$
0.14	447338.9025	37.978695	$1.710542822 \cdot 10^{14}$	$-8.70037801 \cdot 10^{-7}$
0.28	77424.7739	138.204373	$1.285215005 \cdot 10^{14}$	$-12.51374789 \cdot 10^{-7}$
0.42	7908.7368	1270.396848	$1.695937629 \cdot 10^{14}$	$-14.63060830 \cdot 10^{-7}$

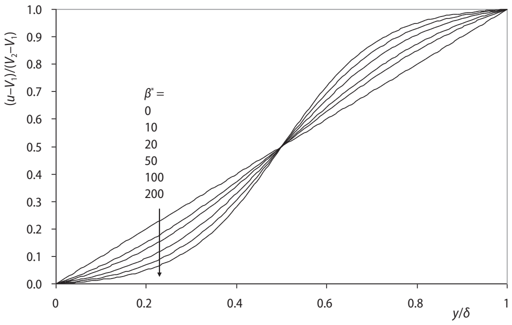

Figure 4. Velocity profile in the film for different values of the viscosity coefficient β^* (particular case $T_1 = T_2 = T_0$)

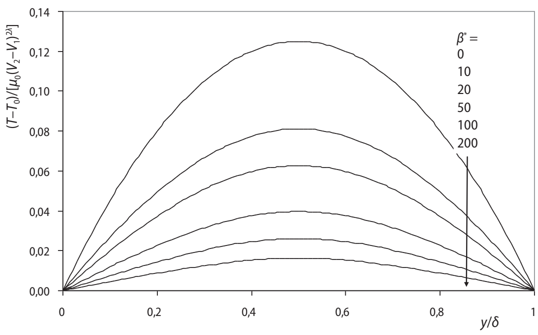

Figure 5. Temperature profile in the film for different values of the viscosity coefficient β^* (particular case $T_1 = T_2 = T_0$)

Figure 6 shows the velocity profile through the film. The velocity gradient is small near the coldest wall (here wall 1, $y/\delta = 0$). This local behaviour is due to the effect of the viscosity which is high for small temperatures. This behaviour is inversed near the hottest wall (here wall 2, $y/\delta = 1$) also due to fact that the viscosity is small for high temperatures.

The temperature profile is given by fig. 7. We note that the maximum of temperature is moved from the centre to the hottest wall (wall 2) when the value of β increases. There is an asymptotic behaviour for which the temperature becomes linear across the film when the value of β is high.

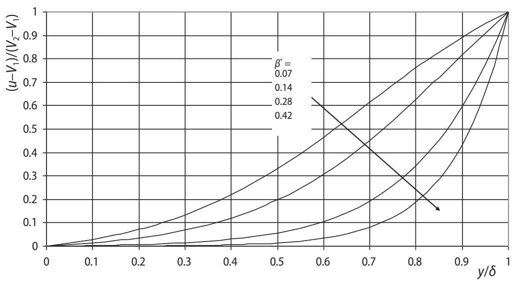

Figure 6. Velocity profiles for cases with $T_1 \neq T_2$ (β = 0.07, 0.14, 0.28, 0.42 K^{-1})

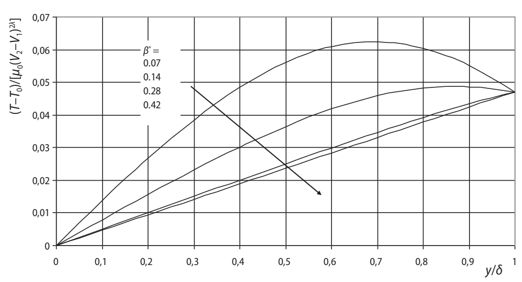

Figure 7. Temperature profiles for cases with $T_1 \neq T_2$ (β = 0.07, 0.14, 0.28, 0.42 K^{-1})

Conclusion

In this paper an analytical solution is proposed to determine the velocity and the temperature profiles in a thin lubricant film separating two moving solids. The dynamic viscosity is considered dependant of the local temperature. The governing equations are then non-linears. The proposed solution is easy to use and allows to investigate several situations of lubrication.

The comparison of results given by the analytical solution to those obtained by a numerical solving shows an excellent agreement.

The results show that the dependence of the viscosity to the temperature has an important effect on the behaviour of the velocity and the temperature profiles in the lubricant film.

Nomenclature

c – specific heat, [Jkg^{-1}K^{-1}]
L – length, [m]
P – pressure, [Pa]
T – temperature, [K] or [ºC]
u – local velocity, [ms^{-1}]
V – velocity of the wall, [ms^{-1}]
x, y – Cartesian co-ordinates, [m]
Δx, Δy – space steps, [m]

Greek symbols

β – viscosity coefficient, [K^{-1}]
δ – lubricant thickness, [m]
λ – thermal conductivity, [Wm^{-1}K^{-1}]
μ – dynamic viscosity, [kgm^{-1}s^{-1}]
ρ – density, [kgm^{-3}]
ω – relaxation coefficient, [–]

Subscripts

0, 1, – reference, solid 1,
2, e – solid 2 and entrance, respectively
i, j – meshes indexes

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