RECONSTRUCTION OF THE THERMAL CONDUCTIVITY COEFFICIENT IN THE SPACE FRACTIONAL HEAT CONDUCTION EQUATION

by

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In this paper an inverse problem for the space fractional heat conduction equation is investigated. Firstly, we describe the approximate solution of the direct problem. Secondly, for the inverse problem part, we define the functional illustrating the error of approximate solution. To recover the thermal conductivity coefficient we need to minimize this functional. In order to minimize this functional the real ant colony optimization algorithm is used. In the model we apply the Riemann-Liouville fractional derivative. The paper presents also some examples to illustrate the accuracy and stability of the presented algorithm.

Key words: *inverse problem, thermal conductivity coefficient, identification, fractional derivative, space fractional heat conduction equation*

Introduction

The inverse problems are some of the most important problems in mathematics. By solving these problems, we can restore parameters of the model basing on some observations produced by the object. An example of the inverse problem is the restoration of parameters in the heat conduction equation. In papers [1-3] there are presented the methods developed for solving the 2-D inverse Stefan problem and the 3-D inverse continuous casting process.

By using the fractional derivatives, we can describe many types of the phenomena in physics, mechanics, and control theory [4-7]. For example, the fractional heat conduction equation better describes the distribution of temperature in porous media, than the classical heat conduction equation [8].

Hristov [9] describes an approximate analytical solution of the spatial-fractional partial differential diffusion equation. This equation describes the phenomenon of super diffusion. In model, considered in this paper, the Riemann-Liouville fractional derivative in space is used. Hristov [10] also considers the transient heat diffusion equation, moreover, a new time-fractional derivative with a non-singular smooth exponential kernel is presented there.

In this paper, firstly, we describe the numerical solution of the space fractional heat conduction equation with Neumann and Robin boundary condition. The fractional derivative, used in considered model, is the Riemann-Liouville fractional derivative. The numerical solution, in case of Dirichlet boundary condition, is presented in [11]. Brociek [12] describes the numerical solution for the time fractional heat conduction equation with mixed boundary condition and with the Caputo derivative.

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The inverse problem, considered in this paper, consists in reconstructing the thermal conductivity coefficient appearing in the space factional heat conduction equation. Authors [13-16] investigate various types of the fractional heat conduction inverse problems with the time fractional Caputo derivative. For example, Zhang [14] introduces an operator and, by using it, he presents an iteration algorithm to reconstruct the time dependent thermal conductivity coefficient. Brociek and Slota [15] present an algorithm to recover the heat transfer coefficient occurring in the third kind boundary condition. The additional information was given there by the measurements of temperature inside the domain. Next, the functional defining the error of approximate solution is created. The authors minimize this functional by using the Nelder-Mead method in order to recover the heat transfer coefficient.

Tatar *et al.* [17] deal with the inverse problem for 1-D space-time fractional diffusion equation. In the considered model the source term is recovered and the proposed numerical solution is based on discretization of the minimization problem.

In order to solve the inverse problem, considered in the current paper, a functional defining the error of approximate solution is created. By minimizing this functional, we recover the thermal conductivity coefficient. To minimize this functional we use the real ant colony optimization (RealACO) algorithm [18]. This algorithm belongs to the group of swarm intelligent algorithms having in recent times a wide range of application [19-21].

Formulation of the problem

In this paper, we consider the following space fractional heat conduction equation:

$$c\rho\frac{\partial u(x,t)}{\partial t}=\lambda(x)\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}} \tag{1}$$

This equation is defined in region $D=\{(x,t):x\in[a,b],\ t\in[0,T)\}$ and c, ρ, λ denote the specific heat, density, and thermal conductivity coefficient, respectively. To complete the previous equation, we add the initial condition:

$$u(x,0)=f(x),\quad x\in[a,b], \tag{2}$$

and boundary conditions of the second and third kind:

$$-\lambda(a)\frac{\partial u(a,t)}{\partial x}=q(t),\quad t\in(0,T) \tag{3}$$

$$-\lambda(b)\frac{\partial u(b,t)}{\partial x}=h(t)\left[u(b,t)-u^{\infty}\right],\quad t\in(0,T) \tag{4}$$

where q is the heat flux, h – the heat transfer coefficient, and u^{∞} – the ambient temperature. In model described by eqs. (1)-(4), according to the spatial variable, we use the fractional Riemann-Liouville derivative [22]:

$$\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}=\frac{1}{\Gamma(n-\alpha)}\frac{\partial^{n}}{\partial x^{n}}\int_{a}^{x}u(s,t)(x-s)^{n-1-\alpha}\mathrm{d}s \tag{5}$$

where Γ is the gamma function [23] and $\alpha\in(n-1,n]$. In our case $1<\alpha<2$ which means that eq. (1) describes the process of super diffusion. For $\alpha=2$, we obtain the equation with classical derivative.

In described model we intend to reconstruct the thermal conductivity coefficient, λ, depending on n parameters a_i, $i = 1, 2, ..., n$. Additional information for the inverse problem will be given by the measurements of temperature at the selected points inside region D:

$$u(x_i, t_j) = \hat{U}_{ij}, \qquad i = 1,2,\ldots,N_1,\ j = 1,2,\ldots,N_2 \tag{6}$$

where N_1 is the number of sensors and N_2 means the number of measurements at each sensor. This information is called the input data (for inverse problem). Solving the direct problem for the fixed value of λ (fixed values of a_i), we obtain the numerical solution of eqs. (1)-(4), that is the approximate values of temperature at the selected points $(x_i,t_j) \in D$. These values will be denoted by $U_{ij}(\lambda)$. Next, we create the functional defining the error of approximate solution:

$$F(\lambda) = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\left[U_{ij}(\lambda) - \hat{U}_{ij}\right]^2 \tag{7}$$

In order to recover the thermal conductivity coefficient, λ, we need to minimize functional (7).

Direct problem

In this section, we describe the numerical solution of direct problem defined by eqs. (1)-(4). In the process of reconstructing parameter λ, it is required to solve the direct problem many times. To solve the direct problem, we used the implicit finite difference scheme. In order to do that, we create the following grid:

$$S = \left\{(x_i, t_k): x_i = a + i\Delta x,\ t_k = k\Delta t,\ i = 0,1,\ldots,N,\ k = 0,1,\ldots,M\right\} \tag{8}$$

where $N \times M$ are the grid size with steps $\Delta x = (b - a)/N$, $\Delta t = T/M$. To approximate the Riemann-Liouville fractional derivative, we apply the Grunwald formula [24]:

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(-\alpha)}\lim_{N\to\infty}\frac{1}{r^\alpha}\sum_{j=0}^{N}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)}u\left[x-(j-1)r,t\right] \tag{9}$$

where $r = (x - a)/N$.

Using the approximation of fractional derivative (9) and the implicit finite difference method, we obtain the differential equations:

$$-\omega_{\alpha,0}\frac{\lambda_i\Delta t}{c\rho(\Delta x)^\alpha}U_{i+1}^{k+1} + \left[1 - \omega_{\alpha,1}\frac{\lambda_i\Delta t}{c\rho(\Delta x)^\alpha}\right]U_i^{k+1} - \omega_{\alpha,2}\frac{\lambda_i\Delta t}{c\rho(\Delta x)^\alpha}U_{i-1}^{k+1} -$$

$$-\omega_{\alpha,0}\frac{\lambda_i\Delta t}{c\rho(\Delta x)^\alpha}\sum_{j=3}^{i+1}\omega_{\alpha,k}U_{i-j+1}^{k+1} = U_i^k \tag{10}$$

where $U_i^k \approx u(x_i, t_k)$, $\lambda_i = \lambda(x_i)$, and $\omega_{\alpha,j} = \Gamma(j-\alpha)/\Gamma(-\alpha)\Gamma(j+1)$. The Neumann and Robin boundary conditions are approximated by formulas:

$$-\lambda_0\left(\frac{-U_2^{k+1} + 4U_1^{k+1} - 3U_0^{k+1}}{2\Delta x}\right) = q^{k+1} \tag{11}$$

$$-\lambda_0\left(\frac{U_{N-2}^{k+1} - 4U_{N-1}^{k+1} + 3U_N^{k+1}}{2\Delta x}\right) = h^{k+1}\left(U_N^{k+1} - u^\infty\right) \tag{12}$$

By solving system of eqs. (10)-(12), we obtain the approximate values of sought function u in the grid points.

Inverse problem

Let us present in this section the algorithm applied for solving the inverse problem. To reconstruct the thermal conductivity coefficient, λ, we need to minimize functional (7). For this purpose, we used the parallel version of the RealACO algorithm. It is an heuristic algorithm inspired by the behaviour of ant swarms in nature. To describe this algorithm, we use the following notation: F – minimized function, n – dimension (number of variables), nT – number of threads, M = nTp – number of ants, I – number of iteration, L – number of pheromone spots (solutions), and q, ξ – parameters of the algorithm.

Now we present the successive steps of the algorithm.

Initialization of the algorithm

(1) Set the input parameters of the algorithm: L, M, I, nT, q, and ξ.
(2) Generate, in random way, L pheromone spots (vectors being the solutions). Assign them to set T_0 (starting archive).
(3) Calculate the value of function F for each pheromone spot (solution). Sort the solutions in archive T_0 from the best one to the worst one.

Iterative process

(4) Assign the probabilities to the pheromone spots (solutions) according to the formula:

$$p_l = \frac{\omega_l}{\sum_{l=1}^{L} \omega_l}, \quad l = 1, 2, \ldots, L$$

where weights ω_l are associated with l-th solution and defined by formula:

$$\omega_l = \frac{1}{qL\sqrt{2\pi}} \mathrm{e}^{\frac{-(l-1)^2}{2q^2L^2}}$$

(5) Ant chooses l-th solution with probability p_l.
(6) Ant transforms the j-th co-ordinate (j = 1, 2, ..., n) of l-th solution s_j^l by sampling proximity, with the use of Gausian function:

$$g(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

with parameters $\mu = s_j^l$, $\sigma = \xi/(L-1)\sum_{p=1}^{L}\left|s_p^l - s_j^l\right|$, as the probability density function.
(7) Repeat steps 5-6 for each ant. In this way M new solutions (pheromone spots) are obtained.
(8) Divide new solutions on nT groups. Calculate the value of minimized function for each ant. Calculations for each group is performed on a separate thread (parallel computing).
(9) Add to the archive the new solutions, sort the archive according to the quality of solutions. Remove M worst solutions.
(10) Repeat steps 3-9 I times.

Numerical results

In this example we consider eq. (1) with data: T = 500 s, a = 0 m, b = 0.2 m, c = 1000 J/kgK, ρ = 2680 kg/m^3, u^∞ = 50 K, $f(x)$ = 0 K, $h(t)$ = 20t W/m^{2}K, $g(t)$ = 0, and α = 1.8.

We assume that the thermal conductivity coefficient is in the form:

$$\hat{\lambda}(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

We have generated the input data for exact form of parameter $\lambda(x) = 100e^{10x} + 100$ and the grid of size 200 × 1000. We approximated the exact form of λ by polynomial of third degree. In the process of minimizing functional (7) we needed to solve many times the direct problem. The grid used in this process was of size 250 × 500. We also assumed one measurement point $x_p = 0.16$ ($N_1 = 1$) and 500 measurements taken from this point. In order to investigate the influence of measurement errors on the results of reconstruction and the stability of the algorithm, the input data were perturbed by the pseudo-random errors of sizes 1% and 3%.

Functional (7) was minimized with the aid of RealACO algorithm with the following parameters: M = 16, L = 16, I = 40, nT = 4, $a_1 \in$ [40000, 49000], $a_2 \in$ [100, 300], $a_3 \in$ [500, 1500], and $a_4 \in$ [50, 400].

Because the RealACO algorithm is the heuristic algorithm, therefore we repeated the calculations ten times in each discussed case. The error of reconstruction is defined:

$$Error = \sqrt{\frac{\int_a^b \left|\lambda(x) - \hat{\lambda}(x)\right|^2 \mathrm{d}x}{\int_a^b \left|\lambda(x)\right|^2 \mathrm{d}x}} 100\%$$

Table 1. Results of the computation (a_i – reconstructed values of the parameters)

Nosise [%]	a_i	Error [%]
0	190.91 1184.34 292.02 48784.26	1.92
1	183.43 1249.23 300.00 47845.29	2.54
3	189.04 1170.48 189.04 49000.00	2.64

Table 1 presents the results of reconstruction obtained for different noises of input data. The error of reconstruction in each case does not exceed 2.65%. This error may be caused by the fact that exact form of the thermal conductivity coefficient was approximated by polynomial.

In figs. (1)-(3) are presented the plots of reconstructed parameter, $\hat{\lambda}$, and exact parameter, λ, in different cases of input data perturbation. As we can see in

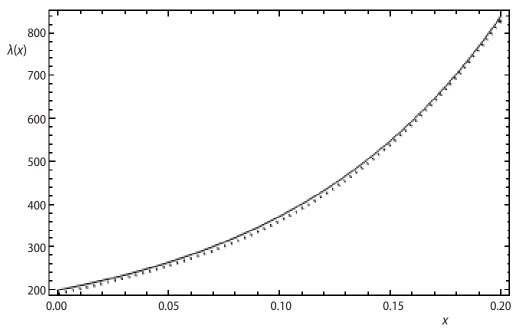

Figure 1. Plots of exact (solid line) and reconstructed (dotted line) thermal conductivity coefficient λ in case of the exact input data

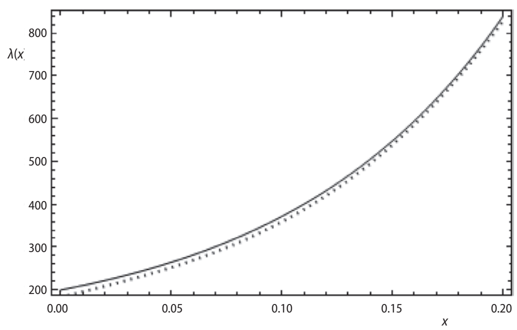

Figure 2. Plots of exact (solid line) and reconstructed (dotted line) thermal conductivity coefficient λ in case of 1% perturbation of the input data

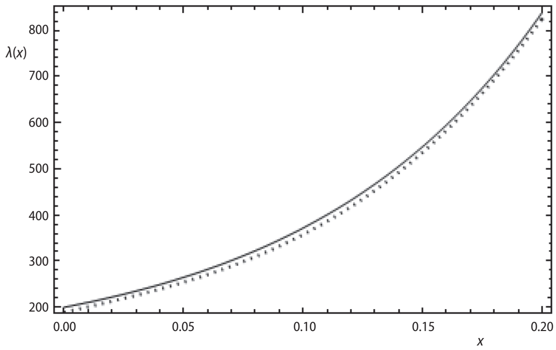

Figure 3. Plots of exact (solid line) and reconstructed (dotted line) thermal conductivity coefficient λ in case of 3% perturbation of the input data

Table 2, Errors of temperature reconstruction in measurement point x_p = 0.16 (Δ_{avg} – average absolute error, Δ_{max} – maximal absolute error, δ_{avg} – average relative error, δ_{max} – maximal relative error)

Noise	0%	1%	3%
Δ_{avg} [K]	$1.80 \cdot 10^{-14}$	$1.27 \cdot 10^{-2}$	$2.19 \cdot 10^{-2}$
Δ_{max} [K]	$4.98 \cdot 10^{-14}$	$4.96 \cdot 10^{-2}$	$7.08 \cdot 10^{-2}$
δ_{avg} [%]	$1.07 \cdot 10^{-13}$	$9.35 \cdot 10^{-2}$	$1.62 \cdot 10^{-1}$
δ_{max} [%]	$4.41 \cdot 10^{-13}$	3.17	4.51

the figures, the thermal conductivity coefficient is reconstructed very well.

One of the main indicators evaluating the obtained results are the errors of temperature reconstruction reconstructed in the measurement point. Table 2 presents such errors of the temperature reconstructed. In case of the exact input data the temperature in measurement point is reconstructed very good. The relative error is small and do not exceed $4.42 \cdot 10^{-13}$. In case of the disturbed input data, the reconstruction errors are larger, but at an acceptable level. For 1% input data perturbation the absolute error is equal to $1.27 \cdot 10^{-2}$ and the relative error is $9.35 \cdot 10^{-2}$%. In case of 3% input data perturbation these errors are equal to $2.19 \cdot 10^{-2}$ and 0.162%, respectively.

Because of the heuristic nature of the used optimization algorithm, in each case of input data the calculations were repeated ten times. Every time, the obtained errors of temperature reconstruction and of thermal conductivity coefficient reconstruction are similar.

Conclusion

In this paper, the fractional heat conduction inverse problem was investigated and the thermal conductivity coefficient was restored. In order to solve the direct problem we used the Grunwald formula to approximate the Riemann-Liouville fractional derivative and we applied the implicit finite difference scheme. Parameter λ was restored as a polynomial in the process of minimizing the proper functional. To minimize this functional the RealACO algorithm was used. Because it is an heuristic algorithm, the calculations were repeated ten times in each discussed case. The obtained results are very good, in case of 1% and 3% input data perturbation the relative error does not exceed $9.36 \cdot 10^{-2}$ % and $1.63 \cdot 10^{-1}$ %, respectively. In case of the exact input data the errors of temperature reconstruction and thermal conductivity coefficient reconstruction are minimal.

Nomenclature

a – beginning of space interval, [m]
b – end of space interval, [m]
c – specific heat, [Jkg^{-1}K^{-1}]
D – domain, [–]
F – minimized fuction, [–]
h – heat transfer coefficient, [Wm^{-2}K^{-1}]
M – dimension of mesh, [–]
N – dimension of mesh, [–]
N_1 – number of sensors, [–]
N_2 – number of measurements, [–]
S – mesh, [–]
T – end of time interval, [s]
t – time, [s]
Δt – time step in mesh, [s]
U_{ij} – computed temperature, [K]
$\hat{U}_{ij}$ – measured temperature (input data), [K]
u^{∞} – ambient temperature, [K]
x – spatial variable, [m]
Δx – space step in mesh, [m]

Greek symbols

α – order of derivative, [–]
Γ – gamma function, [–]
δ – relative error, [%]
Δ – absolute error, [K]
λ – exact thermal conductivity coefficient, [$Wm^{-1}K^{-1}$],
$\hat{\lambda}$ – reconstructed thermal conductivity coefficient, [$Wm^{-3}K^{-1}$]
ρ – density, [kgm^{-3}]

Subscript

avg – average
max – maximal
min – minimal

References

[1] Hetmaniok, E., *et al*., Using the Swarm Intelligence Algorithms in Solution of the Two-Dimensional Inverse Stefan Problem, *Computers & Mathematics with Applications, 69* (2015), 4, pp. 347-361
[2] Hetmaniok, E., *et al*., Experimental Verification of Approximate Solution of the Inverse Stefan Problem Obtained by Applying the Invasive Weed Optimization Algorithm, *Thermal Science, 19* (2015), Suppl. 1, pp. S205-S212
[3] Hetmaniok, E., *et al*., Restoration of the Cooling Conditions in a Three-Dimensional Continuous Casting Process Using Artificial Intelligence Algorithms, *Applied Mathematical Modelling, 39* (2015), 16, pp. 4797-4807
[4] Sabatier, J., *et al*., *Advances of Fractional Calculus, Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, The Netherlands, 2007
[5] Carpinteri, A., Mainardi, F., *Fractal and Fractional Calculus in Continuum Mechanics*, Spinger, New York, USA, 1997
[6] Blasik, M., Numerical Scheme for One-Phase 1D Fractional Stefan Problem Using the Similarity Variable Technique, *Journal of Applied Mathematics and Computational Mechanics, 13*, (2014), 1, pp. 13-21
[7] Blasik, M., Klimek, M., Numerical Solution of the One Phase 1D Fractional Stefan Problem Using the Front Fixing Method, *Mathematical Methods in the Applied Sciences, 38* (2014), 15, pp. 3214-3228
[8] Obraczka, A., Kowalski, J., Heat Transfer Modeling in Ceramic Materials Using Fractional Order Equations, in: *Advances in the Theory and Applications of Non-integer Order Systems,* Lecture Notes in Electrical Engineering, Vol. *257,* 2013, pp. 221-229
[9] Hristov, J., An Approximate Solution to the Transient Space-Fractional Diffusion Equation: Integral-Balance Approach, Optimization Problems and Analyzes, *Thermal Science,* (2016), On-line first, doi:10.2298/TSCI160113075H
[10] Hristov, J., Transient Heat Diffusion with a Non-Singular Fading Memory from the Cattaneo Constitutive Equation with Jeffrey's Kernel to the Caputo-Fabrizio Time-Fractional Derivative, *Thermal Science, 20* (2016), 2, pp. 765-770
[11] Tadjeran, Ch., *et al*., A Second-Order Accurate Numerical Approximation for the Fractional Diffusion Equation, *Journal of Computational Physics, 213* (2006), 1, pp. 205-213
[12] Brociek, R., Implicite Finite Difference Method for Time Fractional Heat Equation with Mixed Boundary Conditions, *Zeszyty Naukowe Politechniki Slaskiej, Matematyka Stosowana, 4* (2014), pp. 73-87
[13] Liu, J. J., *et al*., On the Reconstruction of Unknown Time Dependent Boundary Surces for Time Fractional Diffusion Process by Distributing Measurement, *Inverse Problems 32* (2016), 1, 015009
[14] Zhang, Z., An Undetermined Coefficient Problem for a Fractional Diffusion Equation, *Inverse Problems, 32* (2016), 1, pp. 015011-015042
[15] Brociek, R., Slota, D., Reconstruction of the Boundary Condition for the Heat Conduction Equation of Fractional Order, *Thermal Science, 19* (2015), Suppl. 1, pp. S35-S42
[16] Brociek, R., *et al*., Reconstruction of the Thermal Conductivity Coefficient in the Time Fractional Diffusion Equation, in: *Advances in Modeling and Control of Non-Integer-Order Systems* (Eds. K. J. Latawiec, *et. al*.), Lecture Notes in Electrical Engineering, Vol. 320, 2015, pp. 239-247
[17] Tatar, S., *et al*., Determination of an Unknown Source Term in a Space-Time Fractional Diffusion Equation, *Journal of Fractional Calculus and Applications, 6* (2015), 1, pp. 83-90
[18] Socha, K., Dorigo, M., Ant Colony Optimization for Continuous Domains, *Europan Journal of Operational Research, 185* (2008), 3, pp. 1155-1173
[19] Hetmaniok, E., Solution of the Two-Dimensional Inverse Problem of the Binary Alloy Solidification by Applying the Ant Colony Optimization Algorithm, *International Communications in Heat and Mass Transfer, 67* (2015), Oct., pp. 39-45

[20] Brociek, R., Slota, D., Application of Intelligent Algorithm to Solve the Fractional Heat Conduction Inverse Problem, in: *Information and Software Technologies* (Eds. G. Dregvaite, *et. al*.), Communications in Computer and Information Science, Vol. 538, 2015, Springer, New York, USA, pp. 356-365
[21] Wozniak M., *et al*., Is the Colony of Ants Able to Recognize Graphic Objects? in: *Information and Software Technologies* (Eds. G. Dregvaite, *et. al*.), Communications in Computer and Information Science, Vol. 538, 2015, Springer, New York, USA, pp. 376-387
[22] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, Cal., USA, 1999
[23] Rabsztyn, Sz., *et al*., *Gamma and Beta Functions,* (in Polish), Wydawnictwo Politechniki Slaskiej, Gliwice, Poland, Vol. 1-2, 2012
[24] Meerschaert, M. M., Tadjeran, Ch., Finite Difference Approximations for Fractional Advection-Dispersion Flow Equations, *Journal of Computational and Applied Mathematics, 172* (2006), 1, pp. 65-77

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