A NEW TECHNOLOGY FOR SOLVING DIFFUSION AND HEAT EQUATIONS

by

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In this paper, a new technology combing the variational iterative method and an integral transform similar to Sumudu transform is proposed for the first time for solutions of diffusion and heat equations. The method is accurate and efficient in development of approximate solutions for the partial differential equations.

Key words: *heat transfer, diffusion equation, heat equation, integral transform, approximate solution*

Introduction

The classical heat conduction has been discussed in the literature more than in two centuries with integer [1-3] of non-local fractional [4-6], and local fractional [7-9] operators. Analytical solutions are classical approach in solution transient heat conduction problems [1-3] and recently efficient approximate techniques to non-linear (with non-linear diffusion coefficients) were developed [10-12].

The variational iteration method (VIM) proposed by He [13] was used widely to solve linear and non-linear heat-transfer problems [14] both direct and inverse [12]. This idea of this technology was extended to a solution methodology compromising its basic idea with integral transform methods, such Laplace [15] and Sumudu [16]. In the context, of the integral transforms applied, we mention a new integral transform resembling the Sumudu transform (see more details in [17] mainly addressed to solution of heat conduction problems.

The present paper addresses a new technology combining the VIM and an integral transform similar to the Sumudu transform to solve diffusion and heat equations.

Mathematical models

Following the Fourier (Fick) law, the rate heat energy per unit area, *i. e.* the heat flux $\vec{\mathrm{q}}(x,y,z,t)$ is proportional to the gradient $\nabla\phi(x,y,z,t)$ [1] with the thermal conductivity (TC) κ as a transport coefficient:

$$\vec{\mathrm{q}}(x,y,z,t) = -\kappa\nabla\phi(x,y,z,t) \tag{1a}$$

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or equivalently:

$$\vec{q}(x,y,z,t) = -\kappa\left[\frac{\partial\phi(x,y,z,t)}{\partial x}\vec{i} + \frac{\partial\phi(x,y,z,t)}{\partial y}\vec{j} + \frac{\partial\phi(x,y,z,t)}{\partial z}\vec{k}\right] \tag{1b}$$

In the case of a volumetric heat source $g(x, y, z, t)$ the 3-D version of the model (1) is [2]:

$$\kappa\nabla^2\phi(x,y,z,t) - \rho c\frac{\partial\phi(x,y,z,t)}{\partial t} = g(x,y,z,t) \tag{2a}$$

or equivalently:

$$\kappa\left[\frac{\partial^2\phi(x,y,z,t)}{\partial x^2} + \frac{\partial^2\phi(x,y,z,t)}{\partial y^2} + \frac{\partial^2\phi(x,y,z,t)}{\partial z^2}\right] - \rho c\frac{\partial\phi(x,y,z,t)}{\partial t} = g(x,y,z,t) \tag{2b}$$

where ρ is the mass density (MD), and c is the specific heat capacity (SHC).

Applying the energy balance and continuity equations the 3-D diffusion equation of heat reads [3]:

$$\alpha\nabla^2\phi(x,y,z,t) - \frac{\partial\phi(x,y,z,t)}{\partial t} = 0 \tag{3a}$$

or equivalently:

$$\alpha\left[\frac{\partial^2\phi(x,y,z,t)}{\partial x^2} + \frac{\partial^2\phi(x,y,z,t)}{\partial y^2} + \frac{\partial^2\phi(x,y,z,t)}{\partial z^2}\right] - \frac{\partial\phi(x,y,z,t)}{\partial t} = 0 \tag{3b}$$

where α is the thermal diffusivity (TD).

The 1-D version of eq. (2a) is:

$$\alpha\frac{\partial^2\phi(x,t)}{\partial x^2} - \frac{\partial\phi(x,t)}{\partial t} = h(x,t), \quad (x,t)\in(0,L)\times(0,\infty) \tag{4a}$$

and the diffusion equation in 1-D conduction heat transfer from eq. (3a):

$$\alpha\frac{\partial^2\phi(x,t)}{\partial x^2} - \frac{\partial\phi(x,t)}{\partial t} = 0, \quad (x,t)\in(0,L)\times(0,\infty), \quad h(x,t) = \frac{g(x,t)}{\rho c} \tag{4b}$$

subjected to initial and boundary conditions:

$$\phi(x,0) = \theta(x) \tag{4c}$$

$$\frac{\partial\phi(0,t)}{\partial x} = \Upsilon_1(t), \tag{4d}$$

$$\frac{\partial\phi(L,t)}{\partial x} = \Upsilon_2(t) \tag{4e}$$

Analysis of the method compromising VIM and integral transform

For the seek of clarity in the explanations, we consider the following differential equation in the operator form:

$$\Lambda\phi + \Xi\phi = h \tag{5}$$

where $\Lambda = -\partial/\partial t$ and $\Xi = \alpha\partial^2/\partial x^2$.

Applying the VIM [13], the functional reads:

$$\phi_{n+1}(x,t) = \phi_n(x,t) + \int_0^t \lambda(t-\tau)\left[\Lambda\phi_n(x,\tau) - \Xi\phi_n(x,\tau) - h(x,\tau)\right]\mathrm{d}\tau \tag{6}$$

Further, applying the integral transform to eq. (6) we get:

$$\begin{aligned}\phi_{n+1}(x,\varpi) &= \phi_n(x,\varpi) + \mathrm{Y}\left\{\int_0^t \lambda(t-\tau)\left[\Lambda\phi_n(x,\tau) - \Xi\phi_n(x,\tau) - h(x,\tau)\right]\mathrm{d}\tau\right\} = \\ &= \phi_n(x,\varpi) + \mathrm{Y}\{\lambda(t)\}\mathrm{Y}\{[\Lambda\phi_n(x,t) - \Xi\phi_n(x,t) - h(x,t)]\}\end{aligned} \tag{7}$$

Considering the variation of eq. (7) with respect to $\phi_n(x,\varpi)$, we have:

$$\delta\phi_{n+1}(x,\varpi) = \delta\phi_n(x,\varpi) + \delta\left[\mathrm{Y}\{\lambda(t)\}\mathrm{Y}\{[\Lambda\phi_n(x,t) - \Xi\phi_n(x,t) - h(x,t)]\}\right] = 0 \tag{8}$$

From the intermediate result of eq. (8) we receive that:

$$\begin{aligned}\delta\phi_{n+1}(x,\varpi) &= 1 + \mathrm{Y}\{\lambda(t)\}\delta\left[\mathrm{Y}\{[\Lambda\phi_n(x,t) - \Xi\phi_n(x,t) - h(x,t)]\}\right] = \\ &= 1 + \lambda(\varpi)\delta\{\Lambda\phi_n(x,t)\} = \\ &= 1 + \lambda(\varpi)\delta\left[\frac{1}{\varpi}\phi_n(x,\varpi) - \varpi\phi_n(x,0)\right] = \\ &= 1 + \frac{1}{\varpi}\lambda(\varpi) = \\ &= 0\end{aligned} \tag{9}$$

Thus, finally one obtains:

$$\lambda(\varpi) = -\varpi \tag{10}$$

Therefore, we developed an iteration algorithm applying the integral operator from eqs.(6) and (10), namely:

$$\phi_{n+1}(x,\varpi) = \phi_n(x,\varpi) + \varpi\mathrm{Y}\{\Lambda\phi_n(x,t)\} + \varpi\left[-\Xi\phi_n(x,\varpi) - h(x,\varpi)\right] \tag{11}$$

Consequently, eq. (11) allows obtaining the integral transform solution in the form:

$$\phi(x,\varpi) = \lim_{n\to\infty}\phi_n(x,\varpi) \tag{12}$$

which reduces to:

$$\phi(x,t) = \mathrm{Y}^{-1}\left\{\lim_{n\to\infty}\phi_n(x,\varpi)\right\} \tag{13}$$

Examples of approximate solutions for diffusion and heat problems

Example 1

Let us consider the 1-D diffusion eq. (4b) with the initial-boundary value conditions:

$$\phi(x,0)=\exp(x),\quad \frac{\partial\phi(0,t)}{\partial x}=\exp(\alpha t),\quad \frac{\partial\phi(L,t)}{\partial x}=\exp(L)\exp(\alpha t) \tag{14a,b,c}$$

From eq. (11), we can construct the following iterative algorithm with the integral operator:

$$\phi_{n+1}(x,\varpi)=\phi_n(x,\varpi)-\varpi\mathrm{Y}\left\{\frac{\partial\phi_n(x,t)}{\partial t}\right\}+\varpi\alpha\frac{\partial^2\phi_n(x,\varpi)}{\partial x^2} \tag{15a}$$

subjected to the initial value condition:

$$\phi_0(x,\varpi)=\mathrm{Y}\{\phi(x,0)\}=\varpi\exp(x) \tag{15b}$$

Thus, we have:

$$\phi_1(x,\varpi)=\varpi\exp(x)(1+\varpi\alpha) \tag{15c}$$

$$\phi_2(x,\varpi)=\varpi\exp(x)\left(1+\varpi\alpha+\varpi^2\alpha^2\right) \tag{15d}$$

$$\phi_3(x,\varpi)=\varpi\exp(x)\left(1+\varpi\alpha+\varpi^2\alpha^2+\varpi^3\alpha^3\right) \tag{15e}$$

and so on.

Therefore, the integral transform solution for the diffusion eq. (4b) with the initial-boundary value conditions in eqs. (14a,b,c) reads:

$$\begin{aligned}\phi(x,t)&=\mathrm{Y}^{-1}\left\{\lim_{n\to\infty}\phi_n(x,\varpi)\right\}=\\&=\mathrm{Y}^{-1}\left\{\varpi\exp(x)\left(1+\varpi\alpha+\varpi^2\alpha^2+\varpi^3\alpha^3+...\right)\right\}=\\&=\exp(x)\exp(\alpha t)\end{aligned} \tag{16}$$

The 3-D graphs corresponding to the cases $\alpha = 1$, $\alpha = 2$, $\alpha = 3$, and $\alpha = 4$, are displayed in fig. 1-4, respectively.

Example 2

As the second example, we consider the heat eq. (4a) with the initial-boundary value conditions:

$$\phi(x,0)=\exp(x),\ \frac{\partial\phi(0,t)}{\partial x}=\exp(\alpha t)-t,\ \frac{\partial\phi(L,t)}{\partial x}=\exp(L)\exp(\alpha t)-t \tag{17a,b,c}$$

where $h(x,t) = 1$.

The iteration algorithm with the integral operator reads:

$$\phi_{n+1}(x,\varpi)=\phi_n(x,\varpi)-\varpi\mathrm{Y}\left\{\frac{\partial\phi_n(x,t)}{\partial t}\right\}+\varpi\alpha\frac{\partial^2\phi_n(x,\varpi)}{\partial x^2}-\varpi^2 \tag{18a}$$

subjected to the initial value condition:

$$\phi_0(x,\varpi)=\mathrm{Y}\{\phi(x,0)\}=\exp(x) \tag{18b}$$

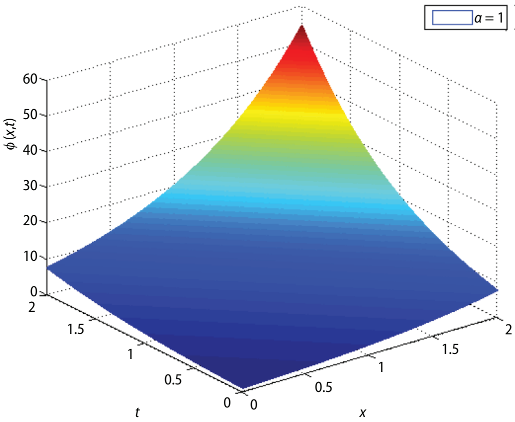

Figure 1. The approximate solution of the diffusion equation for the TD $\alpha = 1$
(*for color image see journal web site*)

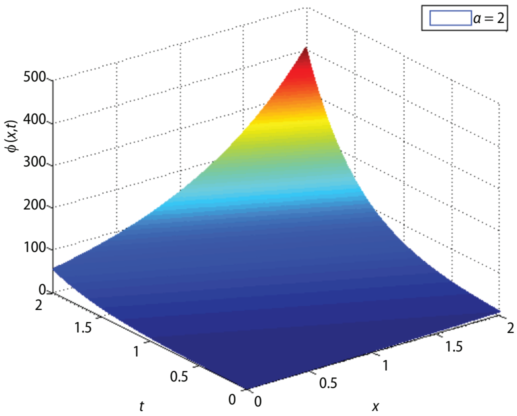

Figure 2. The approximate solution of the diffusion equation for the TD $\alpha = 2$
(*for color image see journal web site*)

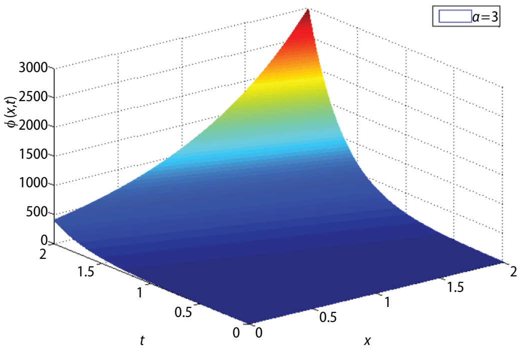

Figure 3. The approximate solution of the diffusion equation for the TD $\alpha = 3$
(*for color image see journal web site*)

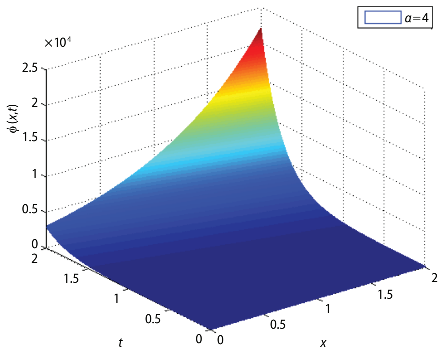

Figure 4. The approximate solution of the diffusion equation for the TD $\alpha = 4$
(*for color image see journal web site*)

From eqs. (18a,b), we have the following (up to $n = 2$ as illustrative example):

$$\phi_1(x,\varpi) = \varpi \exp(x)(1+\varpi\alpha) - \varpi^2 \tag{18c}$$

$$\phi_2(x,\varpi) = \varpi \exp(x)\left(1+\varpi\alpha+\varpi^2\alpha^2\right) - \varpi^2 \tag{18d}$$

$$\phi_3(x,\varpi) = \varpi \exp(x)\left(1+\varpi\alpha+\varpi^2\alpha^2+\varpi^3\alpha^3\right) - \varpi^2 \tag{18e}$$

Therefore, the integral transform solution for the heat eq. (4a) with the initial-boundary conditions (17a,b,c) is:

$$\begin{aligned}\phi(x,t) &= \Upsilon^{-1}\left\{\lim_{n\to\infty}\phi_n(x,\varpi)\right\} = \\ &= \Upsilon^{-1}\left\{\varpi\exp(x)\left(1+\varpi\alpha+\varpi^2\alpha^2+\varpi^3\alpha^3+\ldots\right)-\varpi^2\right\} = \\ &= \exp(x)\exp(\alpha t) - t\end{aligned} \tag{19}$$

The corresponding 3-D plots for the cases $\alpha = 1$, $\alpha = 2$, $\alpha = 3$, and $\alpha = 4$, are presented in figs. 5-8, respectively.

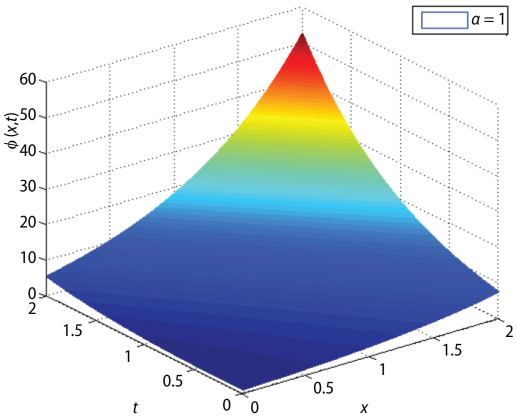

Figure 5. The approximate solution of the diffusion equation for the TD $\alpha = 1$
(*for color image see journal web site*)

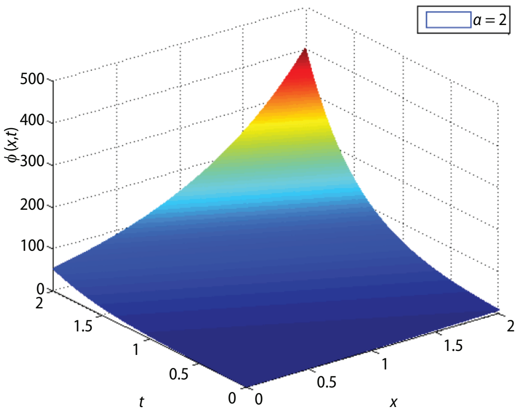

Figure 6. The approximate solution of the diffusion equation for the TD $\alpha = 2$
(*for color image see journal web site*)

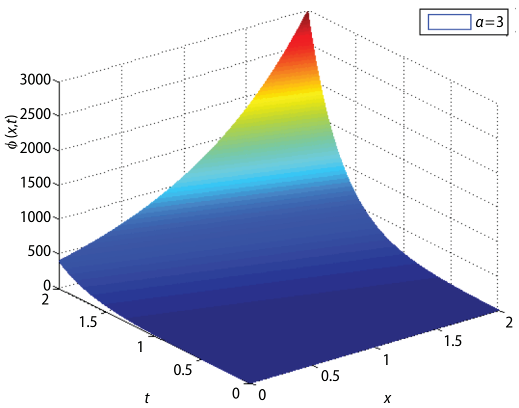

Figure 7. The approximate solution of the diffusion equation for the TD $\alpha = 3$
(*for color image see journal web site*)

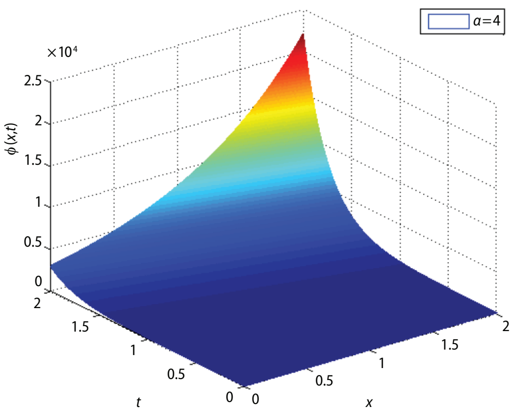

Figure 8. The approximate solution of the diffusion equation for the TD $\alpha = 4$
(*for color image see journal web site*)

Conclusion

The addressed an approximate solution method compromising the VIM and an integral transform similar to Sumudu transform. The solution technology was exemplified by solutions of 1-D transient heat conduction. The proposed method is accurate and efficient and allows straight forwardly develop approximate solutions for the heat-transfer equations by conduction.

Nomenclature

c – the SHC, [$\mathrm{Jkg^{-1}K^{-1}}$]
t – time, [s]
x – space co-ordinate, [m]

Greek symbols

α – the TD, [$\mathrm{m^2s^{-1}}$]
κ – the TC, [$\mathrm{Wm^{-1}K^{-1}}$]
ρ – the MD, [$\mathrm{kgm^{-3}}$]
$\phi(x,t)$ – temperature, [K]
$\phi(\varpi) = \mathrm{Y}\left[\phi(\tau)\right]$ – integral transform, [–]

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Appendix

The integral transform of the function $\phi(t)$ is defined by [17]:

$$\phi(\varpi) = \mathrm{Y}\left[\phi(\tau)\right] = \int_0^\infty \phi(\tau) \mathrm{e}^{-\frac{\tau}{\varpi}} \mathrm{d}\tau, \quad \tau > 0 \tag{A1}$$

The expression (A1) suggests that the integral exists for some ϖ, where $\theta \in (-\tau_1, \tau_2)$ and Y is the integral transform.

The inverse integral transform is given by [17]:

$$\mathrm{Y}^{-1}\left\{\mathrm{Y}\left(\phi(t)\right)\right\} = \mathrm{Y}^{-1}\left\{\phi(\varpi)\right\} = \phi(t) \tag{A2}$$

The properties of the integral transform are briefly outlined [17]:
(R1) Suppose that $\phi_1(\varpi)=\mathrm{Y}[\phi_1(\tau)]$ and $\phi_2(\varpi)=\mathrm{Y}[\phi_2(\tau)]$. Then, we have:

$$\mathrm{Y}\left[a\phi_1(\tau)+b\phi_2(\tau)\right]=a\phi_1(\varpi)+b\phi_2(\varpi) \tag{A3}$$

where a and b are constants.
(R2) Suppose that $\phi(\varpi)=\mathrm{Y}[\phi(\tau)]$ and the derivative of $\phi(\tau)$ is $\phi^{(1)}(\tau)$. Then, we have:

$$\mathrm{Y}\left[\phi^{(1)}(\tau)\right]=\frac{1}{\varpi}\phi(\varpi)-\phi(0) \tag{A4}$$

(R3) Suppose that $\phi(\varpi)=\mathrm{Y}[\phi(\tau)]$ and the integral of $\phi(\tau)$ is $\int_0^\tau \phi(\tau)\mathrm{d}\tau$. Then, we have:

$$\mathrm{Y}\left[\int_0^\tau \phi(\tau)\mathrm{d}\tau\right]=\varpi\phi(\varpi) \tag{A5}$$

(R4) Let μ be a constant. Then we have:

$$\mathrm{Y}\left[\exp(\mu\varpi)\right]=\frac{\varpi}{1-\mu\varpi} \tag{A6}$$

(R5)

$$\mathrm{Y}[1]=\varpi \tag{A7}$$

(R6)

$$\mathrm{Y}[\tau]=\varpi^2 \tag{A8}$$

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