SIMPLE AND ACCURATE CORRELATIONS FOR SOME PROBLEMS OF HEAT CONDUCTION WITH NON-HOMOGENEOUS BOUNDARY CONDITIONS

by

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Original scientific paper
DOI:10.2298/TSCI160411243L

Heat conduction in solids subjected to non-homogenous boundary conditions leads to singularities in terms of heat flux density. That kind of issues can be also encountered in various scientists' fields as electromagnetism, electrostatic, electrochemistry, and mechanics. These problems are difficult to solve by using the classical methods such as integral transforms or separation of variables. These methods lead to solving of dual integral equations or Fredholm integral equations, which are not easy to use.
The present work addresses the calculation of thermal resistance of a finite medium submitted to conjugate surface Neumann and Dirichlet conditions, which are defined by a band-shape heat source and a uniform temperature. The opposite surface is subjected to a homogeneous boundary condition such uniform temperature or insulation.
The proposed solving process is based on simple and accurate correlations that provide the thermal resistance as a function of the ratio of the size of heat source and the depth of the medium. A judicious scale analysis is performed in order to fix the asymptotic behaviour at the limits of the value of the geometric parameter.
The developed correlations are very simple to use and are valid regardless of the values of the defined geometrical parameter.
The performed validations by comparison with numerical modelling demonstrate the relevant agreement of the solutions to address singularity calculation issues.

Key words: *non-homogeneous boundary conditions, analytical modelling, heat conduction with singularities, asymptotic behaviour, simple correlations*

Introduction

Heat conduction problems are usually solved in the literature by using analytical or numerical models. The most popular analytical methods are based on the integral transforms such as Fourier, Hankel, and others [1-5] or integral methods as heat balances [6, 7]. These analytical solutions can be easily deducted when the boundary conditions on the same surface

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are homogeneous. The cases of non-homogeneous boundary conditions are more complex to achieve, for instance, coupled Dirichlet and Neumann on the same surface lead to singularities in terms of heat flux.

These issues are commonly encountered in various scientists' fields as heat transfer, electromagnetism, electrostatic, electrochemistry, and mechanics. Several studies were developed in the literature to resolve these problems [8, 9]. The authors derive solutions under integral form such as the called *Dual integral equations*, or under imbricated *Fredholm integral equations*, which require iterative numerical solving. Collins [10] has studied the potential problem for a circular annulus. He performed analytical developments for the axisymmetric case using a superposition technique. The solutions are given under *Fredholm integral equations*.

Thus some authors addressed the case of Dirichlet condition on an annular disc that involves Bessel's functions. The solutions lead to a triple *dual integral equations* corresponding to the three distinctive parts of the surface: the inner circle, the ring-shaped area, and the outer surface. Cooke [11] has proposed different solutions for the triple integral equations. These solutions lead to *Fedholm integral equations* as that obtained by Collins [10].

Further, Fabrikant [12] considered the Dirichlet problem taking into account the non-axisymmetric. The deducted solution is also under Fredholm's integral equation form. The kernel of this integral equation is non-singular and so can be solved by an iterative method. No general solutions to these problems have been attempted yet.

Recently, two different Dirichlet problems were investigated [13]:

– an annular disc subjected to a Dirichlet condition, with a uniform temperature, when the remaining surfaces are insulated, and
– an isothermal annular disc, with zero temperature, when the inner surface is subjected to a uniform heat flux and the outer surface is insulated.

A judicious approach is proposed to determine the thermal resistance through the annulus. The principle of this method consists on the modelling of the asymptotic behaviours under their most compact form and then the use of a correlation technique to connect them. The provided solutions are compact and their predictions are in excellent agreement with available data of Smythe [8] and Cooke [9].

Moreover, for various problems of conduction involving singularities, it can be more practical to use the conformal mapping method that is based in the Schawrz-Christoffel transformation [14].

The present work focuses in the steady-state calculation of 2-D thermal resistance of a finite medium subjected to a band-shape heating source and a uniform temperature on the same surface. Thus Neumann and Dirichlet conditions are simultaneously applied. The opposite surface is subjected to various homogeneous boundary conditions such as uniform temperature or insulation. Simple and accurate correlations are established that provide the thermal resistance as a function of the ratio of the size of heat source and the depth of the medium. Validations by comparison with numerical modelling show that the promoted solutions are in good agreement.

Studied problem

The study is a finite medium, having a thickness c, shown in fig. 1, which is subjected to a band heat source, φ_c, on the area ($|x| \leq a$, $y = 0$) and an uniform reference temperature $T = 0$ on the remain of the same surface ($|x| > a$, $y = 0$). For the opposite face (*i. e.*, $y = c$) we consider two configurations: (1) the surface is isothermal with the reference temperature $T = 0$ and (2) adiabatic.

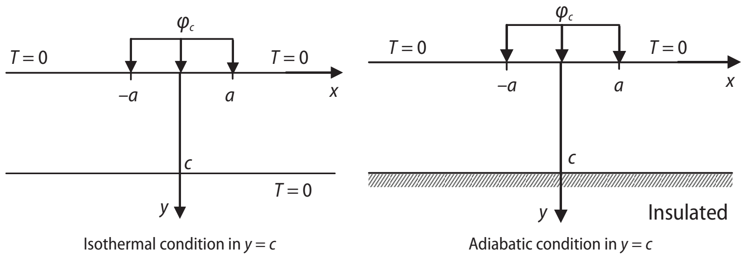

Figure 1. Studied conduction cases

Due to the singularity in terms of heat flux density at the abscissa $|x| = a$, ($y = 0$), the resolution is difficult using the classical methods. Indeed, the heat flux density near these two abscissae becomes infinity.

Several methods can be used to try the solving of this problem such as conformal mapping, green's functions, dual integral equations, Fredholm integral equations, and finally numerically. All these methods reclaim complicated as well as heavy calculations.

In order to calculate the thermal resistance of this problem, we suggest a simple approach, which is based on the analysis of the asymptotic behaviour according the ratio (a/c) and the use of a powerful correlation.

The definition of the thermal resistance is given by:

$$R_c = \frac{T_c - T_0}{\phi_c} \tag{1}$$

where T_c and ϕ_c are the average temperature and heat flux of the contact area, respectively, and T_0 is the reference temperature. Here, we have $\phi_c = 2a\varphi_c$ and $T_0 = 0$.

The aim is to promote user-friendly correlations that can be defined by engineers in various industrial fields without solving the governing equations.

Analytical model developments

Governing equations

The governing equation of both problems described in fig. 1 is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{2}$$

The boundary conditions are given by:

$$\begin{cases} -\lambda \left(\dfrac{\partial T}{\partial y}\right)_{x,y=0} = \varphi_c & (|x| \le a) \\ (T)_{x,y=0} = 0 & (|x| > a) \end{cases} \tag{3}$$

$$T(x, y = c) = 0 \quad \text{or} \quad \left(\frac{\partial T}{\partial y}\right)_{x,y=c} = 0 \tag{4}$$

The next sections present the following elements of the resolution-problem process:

– the asymptotic behaviour according with both boundary conditions of (4) and
– the correlations and their validations.

Analysis of the asymptotic behaviours

– Case of isothermal surface $T(x,y = c) = 0$

The asymptotic behaviour when the depth-source length ratio (c/a) tends to infinity is well known in the literature [15]. In this case, the thermal resistance can be written:

$$R_{c,\infty}^{*} = \lambda R_c (c/a \to \infty) = \frac{\pi}{8} \tag{5}$$

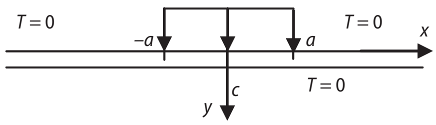

Figure 2. Asymptotic beaviour when $c \to 0$ for isothermal condition at $y = c$

The approach is done when the length ratio is close to zero, corresponding to a very low value of c as displayed in fig. 2.

In this peculiar case, it is easy to note that the thermal resistance is a simple resistance of a wall, which is given by:

$$R_{c,0}^{*} = \lambda R_c (c/a \to 0) = \frac{c}{2a} \tag{6}$$

– Case of adiabatic surface $(\partial T/\partial y)_{x,\,y=c} = 0$

When the depth-source length ratio (c/a) tends to infinity, the solution is the same that the previous case. Then the thermal resistance is given by eq. (5).

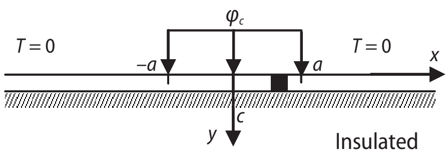

Figure 3. Asymptotic beaviour when $c \to 0$ for adiabatic condition at $y = 0$

At the opposite, when the length ratio is close to zero, we consider the schema given by the fig. 3.

The energy balance for an elementary surface (black cell) between the abscissa x and $x + \mathrm{d}x$, can be written:

$$-\lambda \left(\frac{\mathrm{d}T}{\mathrm{d}x} \right)_x c + \lambda \left(\frac{\mathrm{d}T}{\mathrm{d}x} \right)_{x+\mathrm{d}x} c + \varphi_c \, \mathrm{d}x = 0 \tag{7}$$

That leads to:

$$\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} = -\frac{\varphi_c}{\lambda c} \tag{8}$$

Considering the following boundary conditions:

$$\left(\frac{\mathrm{d}T}{\mathrm{d}x} \right)_{x=0} = 0 \quad \text{(symmetry)}, \quad T(x = a) = 0 \tag{9}$$

The solution of eq. (8) is easy to determine. We obtain:

$$T(x) = \frac{\varphi_c}{2\lambda c} \left(a^2 - x^2 \right) \tag{10}$$

That allows determining the average temperature of the contact area:

$$T_c = \frac{1}{a}\int_0^a T(x)\mathrm{d}x = \frac{1}{a}\frac{\varphi_c}{2\lambda c}\int_0^a \left(a^2 - x^2\right)\mathrm{d}x = \frac{\varphi_c a^2}{3\lambda c} \tag{11}$$

By using the definition of the thermal resistance given by eq. (1), we deduce the asymptotic behaviour for $c/a \to 0$ such:

$$R^*_{c,0} = \lambda R_c(c/a \to 0) = \frac{a}{6c} \tag{12}$$

Correlations

At first, the length ratio a/c is fixed as the key parameter of the relationship.

In order to define a compact expression of R_c, that will be valid regardless of the value of a/c, Churchill and Usagi [16] correlation is used.

In this method the following relationship is derived: $Y = (1 + Z^n)^{1/n}$ where Y and Z are expressed in terms of the solutions for asymptotically large and small values of the independent variable (here, a/c). This correlation is valid for processes with monotonic variation over the entire extent of variation of the parameter (here a/c). This is the case of the studied thermal resistances. The arbitrary exponent n can be evaluated simply by comparison to some known points which are given experimentally or numerically (they are numerically in our study). The exponent n has a positive sign when the function is concave up and a negative sign when the function is concave down.

This kind of correlation is equivalent to write:

$$\left(R_c\right)^n = \left(R_{c,\infty}\right)^n + \left(R_{c,0}\right)^n \tag{13}$$

Then, the correlation form is given:

$$R_c = R_{c,\infty}\left[1 + \left(R_{c,0}/R_{c,\infty}\right)^n\right]^{1/n}, \quad \text{where} \quad R_{c,\infty} = \pi/8 \tag{14}$$

By applying this correlation method to our problem, a dimensionless expression of the thermal resistance is established:

$$R^*_c = \frac{\pi}{8}\left[1 + \left(\frac{8}{\pi}R^*_{c,0}\right)^n\right]^{1/n} \tag{15}$$

According with both boundary condition cases, the specific dimensionless expression can be deducted:

$$\begin{aligned} R^*_c &= \frac{\pi}{8}\left[1 + \left(\frac{4}{\pi}\frac{c}{a}\right)^n\right]^{1/n} \quad \text{for isothermal condition} \quad T(x, y = c) = 0 \\ R^*_c &= \frac{\pi}{8}\left[1 + \left(\frac{4}{3\pi}\frac{a}{c}\right)^m\right]^{1/m} \quad \text{for adiabatic condition} \quad (\partial T/\partial y)_{x,y=c} = 0 \end{aligned} \tag{16}$$

Results and discussions

According with the eqs. (16) when the depth tends toward zero ($c \to 0$), the thermal resistance becomes zero for the isothermal condition and infinity for adiabatic condition, which is evident. So to be coherent with the asymptotic behaviours, the exponent n have to be negative and the exponent m positive.

To determine the values of n and m, we have solved numerically the problem for both boundary conditions considering the symmetry at $x = 0$. We considered only two values of the ratio a/c. To obtain accurate results, the mesh was refined in the vicinity of the singularity ($x = a$).

These two points allow us to deduct the first values of n and m. We have adopted these values and performed the numerical calculation for others values of the ratio a/c. We show that the first values of n and m are valid for all the other ratios a/c. The initial values of n and m can be slightly adjusted to obtain more precise values of R_c by minimization of the relative difference between the correlation and the numerical modelling over the changing in the ratio a/c.

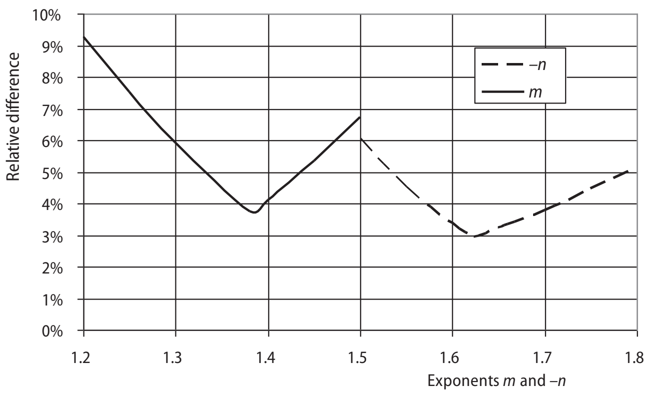

Figure 4. Relative difference between numerical modelling and correlations *vs*. the exponent of each correlation (–*n*: isothermal case, *m*: adiabatic case)

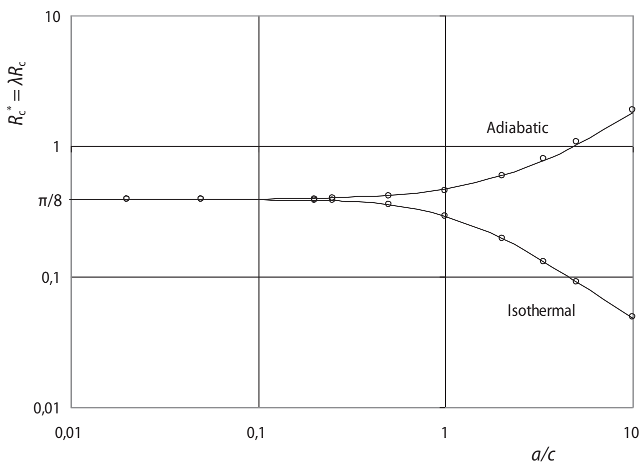

Figure 5. Changes in the thermal resistance as a function of a/c; continues lines (the correlations) points (numerical modelling)

Figure 4 presents the changes in the relative difference between the numerical modelling and correlations as a function the exponents n and m. It is clear that there is an optimal value of the exponent for both studied boundary conditions. The relative difference seems to be highest for the adiabatic case.

The results for the case of isothermal condition are given in fig. 4. The optimal value of the exponent is found to be $n = -1.623$. The relative difference between the correlation and the numerical modelling is near 3%, tab. 1.

Table 2 displays the resultats for the case of adiabatic condition. The optimal value of the exponent is found to be $m = 1.380$. The relative difference between the correlation and the numerical modelling is also near 3.73%.

Figure 5 shows, for both boundary conditions, the changes in the dimensionless thermal resistance as a function of the ratio a/c in a logarithmic scale. The continuous curves are the correlations and the dotted are the numerical results.

Table 1. Comparison between numerical modelling and the correlation (isothermal case)

a/c	R_c^*		Relative difference
	Numerically	Correlation (n = –1.623)	
0.020	0.3903055	0.3924136	0.54%
0.050	0.3917005	0.3914400	0.07%
0.200	0.3857795	0.3811636	1.21%
0.250	0.3823850	0.3764012	1.59%
0.500	0.3577245	0.3475282	2.93%
1.000	0.2927795	0.2857121	2.47%
2.000	0.1955403	0.1963104	0.39%
3.333	0.1300140	0.1333969	2.54%
5.000	0.0910858	0.0938451	2.94%
10	0.0487475	0.0489438	0.40%

Table 2. Comparison between numerical modelling and the correlation (adiabatic case)

a/c	R_c^*		Relative difference
	Numerically	Correlation (m = 1.380)	
0.020	0.3921500	0.3930934	0.24%
0.050	0.3925100	0.3940949	0.40%
0.200	0.3952300	0.4021177	1.71%
0.250	0.3970100	0.4054936	2.09%
0.500	0.4114900	0.4256863	3.33%
1.000	0.4595800	0.4766320	3.58%
2.000	0.5952900	0.6006413	0.89%
3.333	0.8013500	0.7878856	1.71%
5.000	1.0720300	1.0380249	3.28%
10	1.8962900	1.8280486	3.73%

Both curves start from $R_c^* = \pi/8$ which is the common value for $a/c \to 0$ and then separate to move towards zero for the isothermal case, and infinity for the adiabatic one.

The tendencies when $a/c \to \infty$ can be derived from the asymptotic behaviours given by eq. (6) for the isothermal case and eq. (12) for the adiabatic one. The slopes of curves are –1/2 and 1/6, respectively.

Conclusions

Some problems of heat conduction involving singularities in term of heat flux density were considered in this paper. These singularities are due to non-homogeneous boundary conditions that are applied simultaneously on the same surface. Simple and accurate correlations were proposed to determine the thermal resistance as a function of a characteristic geometric parameter based on depth-source length ratio.

The proposed correlations were validated by comparison with numerical computations. Their maximum relative differences are less than 4%. The established correlations are designed to be practical for engineers. Indeed they permit a very comprehensive calculation of the thermal resistance without a complicated numerical computing to carefully account the singularities phenomena.

Moreover, the present analysis can be easily extended to other calculation domains such as the capacitance for electric problems or the current for electrochemistry problems.

Furthermore, the developed correlations are relevant over all the range of variation of the geometric parameter without any restriction.

Nomenclature

a – half-width of the heat source, [m]
c – wall thickness, [m]
R_c – thermal resistance, [mKW^{-1}]
R_c^* – dimensionless thermal resistance, (= λR_c), [–]
T – temperature, [K] or [^{0}C]
x, y – Cartesian co-ordinates, [m]

Greek symbols

λ – thermal conductivity, [Wm^{-1}K^{-1}]
φ – heat flux density, [Wm^{-2}]

Subscripts

c – contact (here the heat source area)
0 – for the reference temperature

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Paper submitted: April 11, 2016
Paper revised: May 10, 2016
Paper accepted: June 25, 2016