ON CHAIN RULE IN FRACTIONAL CALCULUS

by

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Chain rule plays an important role in fractional calculus. There are many definitions of fractional derivative, and this paper shows that the chain rule is invalid for Jumarie's modification of Riemann-Liouville definition.

Key words: *time-fractional Kuramoto-Sivashinsky equation, exact solution, tanh-sech method, nanometer scale*

Introduction

In a recent paper, Sahoo and Saha Ray [1] studied the following time-fractional Kuramoto-Sivashinsky (K-S) equation:

$$D_t^\alpha u \quad auu_x \quad bu_{xx} \quad ku_{xxx} \quad 0 \tag{1}$$

where $0 < \alpha \quad 1$, a, b, k are arbitrary constants, and the fractional derivative D_t^α is described Jumarie's modified Riemann-Liouville sense [2], namely:

$$D_t^\alpha u(x,t) \quad \frac{1}{\Gamma(1 \quad \alpha)} \frac{d}{dt} {}_0^t (t \quad \tau)^{\quad \alpha}[u(x,\tau) \quad u(x,0)]d\tau \tag{2}$$

Equation (1) is very important to study the mechanics of systems at nanometerscale. Sahoo and Saha Ray [1] obtained eight analytical exact solutions by using tanh-sech method with the help of fractional complex transform [3]. Their approach is strongly based on the following Jumarie's simple chain rule:

$$D_t^\alpha f(g(t)) \quad \frac{df(g(t))}{dg} D_t^\alpha g(t) \tag{3}$$

However, some counterexamples have been appeared in literature, see [4-6]. In this paper, the chain rule will be checked again.

Analysis and result

For the reader's convenience, we here list Sahoo's eight solutions [1]:

(s1) $$\Phi_1 \quad \frac{15}{19a}\sqrt{\frac{11}{19k}} b^{3/2}[2 \quad \tanh(\xi) \quad 11\tanh^3(\xi)] \tag{4}$$

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where $$\xi \quad \frac{1}{2}\sqrt{\frac{11b}{19k}}\; x \quad \sqrt{\frac{11}{19k}}\,\frac{30b^{3/2}t^{\alpha}}{19\Gamma(\alpha \quad 1)}$$

(s2) $$\Phi_2 \quad \frac{15}{19a}\sqrt{\frac{11}{19k}}\,b^{3/2}[2 \quad 9\tanh(\xi) \quad 11\tanh^3(\xi)] \tag{5}$$

where $$\xi \quad \frac{1}{2}\sqrt{\frac{11b}{19k}}\; x \quad \sqrt{\frac{11}{19k}}\,\frac{30b^{3/2}t^{\alpha}}{19\Gamma(\alpha \quad 1)}$$

(s3) $$\Phi_3 \quad \frac{15}{19a}\sqrt{\frac{11}{19k}}\,b^{3/2}[2 \quad 9\tanh(\xi) \quad 11\tanh^3(\xi)] \tag{6}$$

where $$\xi \quad \frac{1}{2}\sqrt{\frac{11b}{19k}}\; x \quad \sqrt{\frac{11}{19k}}\,\frac{30b^{3/2}t^{\alpha}}{19\Gamma(\alpha \quad 1)}$$

(s4) $$\Phi_4 \quad \frac{15}{19a}\sqrt{\frac{11}{19k}}\,b^{3/2}[2 \quad 9\tanh(\xi) \quad 11\tanh^3(\xi)] \tag{7}$$

where $$\xi \quad \frac{1}{2}\sqrt{\frac{11b}{19k}}\; x \quad \sqrt{\frac{11}{19k}}\,\frac{30b^{3/2}t^{\alpha}}{19\Gamma(\alpha \quad 1)}$$

(s5) $$\Phi_5 \quad \frac{15}{19a}\sqrt{\frac{1}{19k}}\,b^{3/2}[2 \quad 3\tanh(\xi) \quad \tanh^3(\xi)] \tag{8}$$

where $$\xi \quad \frac{1}{2}\sqrt{\frac{b}{19k}}\; x \quad \sqrt{\frac{1}{19k}}\,\frac{30b^{3/2}t^{\alpha}}{19\Gamma(\alpha \quad 1)}$$

(s6) $$\Phi_6 \quad \frac{15}{19a}\sqrt{\frac{1}{19k}}\,b^{3/2}[2 \quad 3\tanh(\xi) \quad \tanh^3(\xi)] \tag{9}$$

where $$\xi \quad \frac{1}{2}\sqrt{\frac{b}{19k}}\; x \quad \sqrt{\frac{1}{19k}}\,\frac{30b^{3/2}t^{\alpha}}{19\Gamma(\alpha \quad 1)}$$

(s7) $$\Phi_7 \quad \frac{15}{19a}\sqrt{\frac{1}{19k}}\,b^{3/2}[2 \quad 3\tanh(\xi) \quad \tanh^3(\xi)] \tag{10}$$

where $$\xi \quad \frac{1}{2}\sqrt{\frac{b}{19k}}\; x \quad \sqrt{\frac{1}{19k}}\,\frac{30b^{3/2}t^{\alpha}}{19\Gamma(\alpha \quad 1)}$$

(s8) $$\Phi_8 \quad \frac{15}{19a}\sqrt{\frac{1}{19k}}\,b^{3/2}[2 \quad 3\tanh(\xi) \quad \tanh^3(\xi)] \tag{11}$$

where $$\xi \quad \frac{1}{2}\sqrt{\frac{b}{19k}}\; x \quad \sqrt{\frac{1}{19k}}\,\frac{30b^{3/2}t^{\alpha}}{19\Gamma(\alpha \quad 1)}$$

Next, we will prove that these solutions are not true.

By eq. (2), we can re-write the eq. (1):

$$\frac{1}{\Gamma(1 \quad \alpha)}\frac{\mathrm{d}}{\mathrm{d}t}{}_0^t(t \quad \tau)[u(x,\tau) \quad u(x,0)]\,\mathrm{d}\tau \qquad auu_x \quad bu_{xx} \quad ku_{xxx} \tag{12}$$

For simplicity, we choose $a = 1$, $b = 1$, $k = 11/19$, and $\alpha = 0.5$ in eq. (1) for checking the obtained solutions listed previously.

If the function (4) is a solution of the fractional differential eq. (1), then the function:

$$u(x,t) \qquad \frac{30}{19} \quad \frac{135}{19}\tanh \quad \frac{x}{2} \quad \frac{15t^{0.5}}{19\Gamma(1.5)} \qquad \frac{165}{19}\tanh^3 \quad \frac{x}{2} \quad \frac{15t^{0.5}}{19\Gamma(1.5)} \tag{13}$$

satisfies the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_0^t(t \quad \tau)^{0.5}[u(x,\tau) \quad u(x,0)]\mathrm{d}\tau \qquad \Gamma(0.5) \;\; uu_x \quad u_{xx} \quad \frac{11}{19}u_{xxx} \tag{14}$$

Integrating both sides of the eq. (14) with respect to time from 0 to 1, we have:

$${}_0^t(t \quad \tau)^{0.5}[u(x,\tau) \quad u(x,0)]\mathrm{d}\tau \qquad \Gamma(0.5){}_0^t \;\; uu_x \quad u_{xx} \quad \frac{11}{19}u_{xxx} \quad \mathrm{d}t \tag{15}$$

Take $x = 0$ in eq. (15), we have:

$$\begin{aligned} &{}_0^t(t \quad \tau) \;\; ^{0.5} \; \frac{165}{19}\tanh^3 \; \frac{15\sqrt{t}}{19\Gamma(1.5)} \qquad \frac{135}{19}\tanh \; \frac{15\sqrt{t}}{19\Gamma(1.5)} \quad \mathrm{d}t \\ &\qquad \Gamma(0.5){}_0^1 \;\; uu_x \quad u_{xx} \quad \frac{11}{19}u_{xxx}\Big|_{x \;\; 0} \mathrm{d}t \end{aligned} \tag{16}$$

However, by MAPLE software, we obtain that left side of eq. (16) approximately equals –4.26 and right approximately equals –0.25.

Thus, the function (4) is not a solution of the eq. (1). Similarly we can prove that the functions (5)-(11) do not satisfy eq. (1).

Discussion and conclusions

Chain rule is not valid for Jumarie's definition of fractional derivative, and new definitions for fractional derivative are much needed. Some new trends in this direction are lattice fractional derivative [7], discrete fractional derivative [8], and He's fractional derivative [9].

This paper gives an effective way to check the chain rule for fractional calculus.

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