HE'S FRACTIONAL DERIVATIVE FOR NON-LINEAR FRACTIONAL HEAT TRANSFER EQUATION

by

Kang-Le WANG and San-Yang LIU *

[a] School of Mathematics and Statistics, Xidian University, Xi'an, China

Original scientific paper
DOI: 10.2298/TSCI1603793W

This paper adopts He's fractional derivative for non-linear fractional heat transfer equation. The fractional complex transform and He's variational iteration method are used to solve the fractional equation.

Key words: *fractal derivative, variational iteration method, fractional complex transform, fractional heat transfer equation*

Introduction

Recently, fractional derivatives have found many applications in various fields of physical sciences such as heat transform, reaction diffusion, control, and so on. The fractional derivatives have many kinds of definitions. The mostly applied ones are:

(1) Riemann-Liouville definition [1]:

$$D_x^{\alpha}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1} f(t)\,dt \tag{1}$$

(2) Caputo's definition [1]:

$$D_x^{\alpha}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}\frac{d^n f(t)}{dt^n}\,dt \tag{2}$$

(3) Xiao-Jun Yang's definition [2]:

$$D_x^{(\alpha)} f(x_0) = f^{(\alpha)}(x_0) = \left.\frac{d^{\alpha} f(x)}{dx^{\alpha}}\right|_{x=x_0} = \lim_{x\to x_0}\frac{\Delta^{\alpha}[f(x)-f(x_0)]}{(x-x_0)^{\alpha}} \tag{3}$$

where $\Delta^{\alpha}[f(x)-f(x_0)] \cong \Gamma(1+\alpha)\,\Delta[f(x)-f(x_0)]$.

(4) Jumarie's definition [3]:

$$D_x^{\alpha}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}[f(t)-f(0)]\,dt \tag{4}$$

(5) He's fractal derivative [4-6]:

$$\frac{Du}{Dx^{\alpha}} = \Gamma(1+\alpha)\lim_{\Delta x = x_1 - x_2 \to L}\frac{u(x_1)-u(x_2)}{(x_1-x_2)^{\alpha}} \tag{5}$$

where Δx does not tend to zero.

---

* Corresponding author; e-mail: liusanyang@126.com

(6) He's fractional derivative [7-9]:

$$\mathrm{D}_t^{\alpha} f = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1} [f_0(s) - f(s)] \mathrm{d}s \tag{6}$$

where $f_0(t)$ is a known function.

In this paper, we use He's variational iteration method (VIM) [10-13] and fractional complex transform [14-16] to solve the non-linear fractional heat transfer equation. The fractional complex transform was first proposed by He [14-16]. The fractional complex transform can convert fractional differential equation into its differential partner, therefore the VIM can be effectively applied.

We consider the non-linear time-fractional heat transfer equation in the following form [17]:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} - 2u^3 \tag{7}$$

with the following initial condition:

$$u(x,0) = \frac{1+2x}{x^2+x+1} \tag{8}$$

where $\partial^{\alpha}/\partial t^{\alpha}$ is He's fractional derivative defined:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^{\alpha}} \int_{t_0}^{t} (s-t)^{n-\alpha-1} [u_0(s) - u(s)] \mathrm{d}s \tag{9}$$

where $u_0(x,t)$ is the solution of its continuous partner of the problem with the same initial condition of the fractal partner.

There are many analytical methods to solve eq. (7), for examples, homotopy perturbation method [18] and sub-equation method [19]. This paper applies the VIM [10-13] to search for an approximate solution of the equation.

Variational iteration method

Consider the following differential equation:

$$Lu + Nu = g(x) \tag{10}$$

where N is a non-linear operator, L is a linear operator, and $g(x)$ is a homogeneous term.

According to the VIM [10-13], we construct a correct functional for eq. (10):

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \{Lu_n(\xi) + N\hat{u}_n(\xi) - g(\xi)\} \mathrm{d}\xi \tag{11}$$

where λ is a Lagrange multiplier, which can be identified optimally via variational theory. The second term on the right is called the correction, and $\hat{u}_n$ is considered a restricted variation, *i. e.* $\delta \hat{u}_n = 0$.

Numerical application

The first step to solve eq. (7) by VIM is to convert the equation into its differential partner by the fractional complex transform [14-16]:

$$T = \frac{t^{\alpha}}{\Gamma(1+\alpha)} \tag{12}$$

We can easily convert eq. (7) into a differential equation, which is the following form:

$$\frac{\partial u}{\partial T} \quad \frac{\partial^2 u}{\partial x^2} \quad 2u^3 \tag{13}$$

with the following initial condition:

$$u(x,0) \quad \frac{1 \quad 2x}{x^2 \quad x \quad 1} \tag{14}$$

Using the VIM, we have the correct functional:

$$u_{n\ 1}(x,T) \quad u_n(x,T) \quad {}_0^T \lambda \; \frac{\partial u_n(x,\xi)}{\partial \xi} \quad \frac{\partial^2 \hat{u}_n(x,\xi)}{\partial x^2} \quad 2\hat{u}_n(x,\xi)^3 \;\; \mathrm{d}\xi \tag{15}$$

The stationary conditions are given as the following form:

$$1 \quad \lambda \quad 0, \qquad \lambda\big|_{\xi \ \ T} \quad 0 \tag{16}$$

which implies $\lambda = -1$. Therefore, we obtain the following iteration formula:

$$u_{n\ 1}(x,T) \quad u_n(x,T) \quad {}_0^T \; \frac{\partial u_n(x,\xi)}{\partial \xi} \quad \frac{\partial^2 \hat{u}_n(x,\xi)}{\partial x^2} \quad 2\hat{u}_n(x,\xi)^3 \;\; \mathrm{d}\xi \tag{17}$$

We set eq. (14) as the initial approximation $u_0(x, T)$. Then using the iteration formula eq. (17), we obtain the following results:

$$u_1(x,T) \quad \frac{1 \quad 2x}{x^2 \quad x \quad 1} \quad \frac{6(1 \quad 2x)}{x^2 \quad x \quad 1} T$$

$$u_2(x,T) \quad \frac{1 \quad 2x}{x^2 \quad x \quad 1} \quad \frac{6(1 \quad 2x)}{(x^2 \quad x \quad 1)^2} T \quad \frac{36(1 \quad 2x)}{(x^2 \quad x \quad 1)^3} T^2$$

$$u_3(x,T) \quad \frac{1 \quad 2x}{x^2 \quad x \quad 1} \quad \frac{6(1 \quad 2x)}{(x^2 \quad x \quad 1)^2} T \quad \frac{36(1 \quad 2x)}{(x^2 \quad x \quad 1)^3} T^2 \quad \frac{216(1 \quad 2x)}{(x^2 \quad x \quad 1)^4} T^3$$

...

Substituting eq. (12) into previous results, we have:

$$u_0(x,T) \quad \frac{1 \quad 2x}{x^2 \quad x \quad 1}$$

$$u_1(x,T) \quad \frac{1 \quad 2x}{x^2 \quad x \quad 1} \quad \frac{6(1 \quad 2x)}{(x^2 \quad x \quad 1)^2} \frac{t^\alpha}{\Gamma(\alpha \quad 1)}$$

$$u_2(x,T) \quad \frac{1 \quad 2x}{x^2 \quad x \quad 1} \quad \frac{6(1 \quad 2x)}{(x^2 \quad x \quad 1)^2} \frac{t^\alpha}{\Gamma(\alpha \quad 1)} \quad \frac{36(1 \quad 2x)}{(x^2 \quad x \quad 1)^3} \quad \frac{t^\alpha}{\Gamma(\alpha \quad 1)}^2$$

$$u_3(x,T) \quad \frac{1 \quad 2x}{x^2 \quad x \quad 1} \quad \frac{6(1 \quad 2x)}{(x^2 \quad x \quad 1)^2} \frac{t^\alpha}{\Gamma(\alpha \quad 1)} \quad \frac{36(1 \quad 2x)}{(x^2 \quad x \quad 1)^3} \quad \frac{t^\alpha}{\Gamma(\alpha \quad 1)}^2$$

$$\frac{216(1 \quad 2x)}{(x^2 \quad x \quad 1)^4} \quad \frac{t^\alpha}{\Gamma(\alpha \quad 1)}^3$$

...

Therefore, the 4th-order approximate solution of eq. (7) is:

$$\Phi_4(x,t) \quad \frac{1 \quad 2x}{x^2 \quad x \quad 1} \quad \frac{6(1 \quad 2x)}{(x^2 \quad x \quad 1)^2} \frac{t^\alpha}{\Gamma(\alpha \quad 1)} \quad \frac{36(1 \quad 2x)}{(x^2 \quad x \quad 1)^3} \quad \frac{t^\alpha}{\Gamma(\alpha \quad 1)}^2$$

$$\frac{216(1 \quad 2x)}{(x^2 \quad x \quad 1)^4} \quad \frac{t^\alpha}{\Gamma(\alpha \quad 1)}^3 \tag{18}$$

Conclusions

In this paper, we have successfully used fractional complex transform and He's VIM to find the approximate solution of the non-linear fractional heat transfer equation based on He's fractional derivative. The result shows that the proposed method is very efficient, powerful and easy mathematical method for solving the non-linear fractional differential equations in science and engineering.

Acknowledgment

This work is supported by National Nature Science Foundation of China (No.: 61373174).

References

[1] Podlubny, I., *Fractional Differential Equations*, Academic Press, New York, USA, 1999
[2] Yang, X. J., *Advanced Local Fractional Calculus and its Applications*, World Science Publisher, New York, USA, 2012
[3] Jumarie, G., Modified Riemann-Liouville Derivative and Fractional Taylor Series of Non-Differentiable Functions Further Results, *Comp. Math. Appl, 51* (2006), 1, pp. 137-1376
[4] He, J.-H., A New Fractal Derivation, *Thermal Science, 15* (2011), Suppl. 1, pp. S145-S147
[5] Fan, J., He, J.-H., Fractal Derivative Model for Air Permeability in Hierarchic Porous Media, *Abstract and Applied Analysis, 2012* (2012), ID354701
[6] Fan, J., He, J.-H., Biomimic Design of Multi-Scale Fabric with Efficient Heat Transfer Property, *Thermal Science, 16* (2012), 5, pp. 1349-1352
[7] He, J.-H., A Tutorial Review on Fractal Space Time and Fractional Calculus, *Int. J. Theor. Phys, 53* (2014), 11, pp. 3698-3718
[8] Sayevand, K., Pichaghchi, K., Analysis of Non-Linear Fractional KdV Equation Based on He's Fractional Derivative, *Non-Linear Science Letters A., 7* (2016), 3, pp. 77-85
[9] Liu, F. J., *et al.*, He's Fractional Derivative for Heat Conduction in a Fractal Medium Arising in Silkworm Cocoon Hierarchy, *Thermal Science, 19* (2015), 4, pp. 1155-1159
[10] He, J.-H., Asymptotic Methods for Solitary Solutions and Compactons, *Abstract and Applied Analysis, 2012* (2012), ID916793
[11] He, J.-H., Some Asymptotic Methods for Strongly Nonlinear Equations, *Int. J. Mod. Phys. B, 20* (2006), 10, pp. 1141-1199
[12] Liu, J. F., Modified Variational Iteration Method for Varian Boussinesq Equation, *Thermal Science, 19* (2015), 4, pp.1195-1199
[13] Huang, L. L., Matrix Lagrange Multiplier of the VIM, *Journal of Computational Complexity and Applications, 2* (2016), 3, pp. 86-88
[14] He, J.-H., Li, Z. B., Converting Fractional Differential Equations into Partial Differential Equations, *Thermal Science,16* (2012), 2, pp. 331-334
[15] Abdou, M. A., *et al.*, New Exact Solutions of the Space-Time Fractional Non-Linear Transport Equation in Porous Medi, *Non-Linear Science Letters, A., 7* (2016), 3, pp. 86-95
[16] Li, Z. B., *et al.*, Exact Solutions of Time-Fractional Heat Conduction Equation by the Fractional Complex Transform, *Thermal Science, 16* (2012), 2, pp. 335-338
[17] Yan, L. M., Modified Homotopy Perturbation Method Coupled with Laplace Transform for Fractional Heat Transfer and Porous Media Equation, *Thermal Science, 17* (2013), 5, pp. 1409-1414
[18] Zhang, M. F., *et al.*, Efficient Homotopy Perturbation Method for Fractional Nonlinear Equations Using Sumudu Transform, *Thermal Science, 19* (2015), 4, pp. 1167-1171
[19] Ma, H. C., *et al.*, Exact Solutions of Nonlinear Fractional Partial Differential Equations by Fractional Sub-Equation Method, *Thermal Science, 19* (2015), 4, pp. 1239-1244

Paper submitted: December 10, 2015
Paper revised: February 1, 2016
Paper accepted: February 1, 2016