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**A NEW FRACTIONAL DERIVATIVE WITHOUT SINGULAR KERNEL
Application to the Modelling of the Steady Heat Flow**

by

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In this article we propose a new fractional derivative without singular kernel. We consider the potential application for modeling the steady heat-conduction problem. The analytical solution of the fractional-order heat flow is also obtained by means of the Laplace transform.

Key words: *heat conduction, steady heat flow, analytical solution, Laplace transform, fractional derivative without singular kernel*

Introduction

Fractional derivatives with singular kernel [1], namely, the Riemann-Liouville [2, 3], Caputo [4, 5], and other derivatives, see [6-8] and the references therein, have nowadays a wide application in the field of heat-transfer engineering.

More recently, the fractional Caputo-Fabrizio derivative operator without singular kernel was given [1, 9-12]:

$${}^{CF}D_x^{(\nu)}T(x) = \frac{(2-\nu)\mathfrak{Z}(\nu)}{2(1-\nu)} \int_0^x \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right] T^{(1)}(\lambda)d\lambda \quad (1)$$

where $\mathfrak{Z}(\nu)$ is a normalization constant depending on ν ($0 < \nu < 1$).

Following eq. (1), Losada and Nieto [10] suggested the new fractional Caputo-Fabrizio derivative operator [11, 12]:

$${}^{CF}D_x^{(\nu)}T(x) = \frac{1}{1-\nu} \int_0^x \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right] T^{(1)}(\lambda)d\lambda \quad (2)$$

where ν ($0 < \nu < 1$) is a real number and $\mathfrak{Z}(\nu) = 2/(2-\nu)$.

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Equations (1) and (2) represent an extension of the Caputo fractional derivative with singular kernel. However, an analog of the Riemann-Liouville fractional derivative with singular kernel has not yet been formulated. The main aim of the article is to propose a new fractional derivative without singular kernel, which is an extension of the Riemann-Liouville fractional derivative with singular kernel, and to study its application in the modeling of the fractional-order heat flow.

Mathematical tools

The Riemann-Liouville fractional derivative of fractional order ν of the function $T(x)$ is defined [1]:

$${}^{RL}D_{a^+}^{(\nu)}T(x) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dx} \int_a^x \frac{T(\lambda)}{(x-\lambda)^\nu} d\lambda \quad (3)$$

where $a \leq x$ and ν ($0 < \nu < 1$) is a real number.

Replacing the function $1/(x-\lambda)^\nu \Gamma(1+\nu)$ by $\mathfrak{R}(\nu) \exp\{-\nu/(1-\nu)\}(x-\lambda)\}/(1-\nu)$, we obtain a new fractional derivative given by:

$$D_{a^+}^{(\nu)}T(x) = \frac{\mathfrak{R}(\nu)}{1-\nu} \frac{d}{dx} \int_a^x \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right] T(\lambda) d\lambda \quad (4)$$

where $a \leq x$, ν ($0 < \nu < 1$) is a real number, and $\mathfrak{R}(\nu)$ is a normalization function depending on ν such that $\mathfrak{R}(0) = \mathfrak{R}(1) = 1$.

Taking $\psi = 1/\nu - 1$, with $0 < \psi < +\infty$, eq. (4) can be re-written:

$$D_{a^+}^{\left(\frac{1}{\psi+1}\right)}T(x) = \aleph(\psi) \frac{d}{dx} \int_a^x \Pi(\lambda) T(\lambda) d\lambda \quad (5)$$

where $\aleph(\psi) = (\psi+1)\mathfrak{R}[1/(\psi+1)]$, and $\Pi(\lambda) = \exp[-(x-\lambda)/\psi]/\psi$.

With the help of the following approximation to the identity [9, 13]:

$$\lim_{\psi \rightarrow 0} \Pi(\lambda) = \delta(x-\lambda) \quad (6)$$

where $\nu \rightarrow 1$ (or $\psi \rightarrow 0$), eq. (4) becomes:

$$\lim_{\nu \rightarrow 1} D_{a^+}^{(\nu)}T(x) = \lim_{\psi \rightarrow 0} \aleph(\psi) \frac{d}{dx} \int_a^x \Pi(\lambda) T(\lambda) d\lambda = T^{(1)}(x) \quad (7)$$

When $\nu \rightarrow 0$ (or $\psi \rightarrow +\infty$), eq. (4) can be written:

$$\lim_{\nu \rightarrow 0} D_{a^+}^{(\nu)}T(x) = \lim_{\nu \rightarrow 0} \frac{\mathfrak{R}(\nu)}{1-\nu} \frac{d}{dx} \int_a^x \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right] T(\lambda) d\lambda = T(x) \quad (8)$$

Taking the Laplace transform of the new fractional derivative without singular kernel for the parameter $a = 0$, we have:

$$L[D_0^{(\nu)}T(x)] = \frac{\mathfrak{R}(\nu)s}{\nu(1-s) + s} T(s) \quad (9)$$

where $L[\xi(x)] := \int_0^x \exp(-sx)\xi(x)dx = \xi(s)$ represents the Laplace transform of the function $\xi(x)$, [14].

We now consider:

$$T(s) = \left[\frac{\nu}{\Re(\nu)s} + \frac{1-\nu}{\Re(\nu)} \right] \Xi(s) \quad (10)$$

where $D_0^{(\nu)}T(x) = \Xi(x)$ and $L[\Xi(x)] = \Xi(s)$.

Taking the inverse Laplace transform of eq. (10) we obtain:

$$T(x) = \frac{1-\nu}{\Re(\nu)} \Xi(x) + \frac{\nu}{\Re(\nu)} \int_0^x \Xi(x) dx, \quad x > 0, \quad 0 < \nu < 1 \quad (11)$$

If $0 < \nu < 1$ and $\Re(\nu) = 1$, then eqs. (4) and (11) can be written:

$$*D_{a^+}^{(\nu)}T(x) = \frac{1}{1-\nu} \frac{d}{dx} \int_a^x \exp \left[-\frac{\nu}{1-\nu} (x-\lambda) \right] T(\lambda) d\lambda \quad (12)$$

and

$$T(x) = (1-\nu)\Xi(x) + \nu \int_0^x \Xi(x) dx, \quad x > 0, \quad 0 < \nu < 1 \quad (13)$$

respectively.

Modelling the fractional-order steady heat flow

The fractional-order Fourier law in 1-D case is suggested:

$$KD_0^{(\nu)}T(x) = -H(x) \quad (14)$$

where K is the thermal conductivity of the material and $H(x)$ – the heat flux density.

The heat flow of the fractional-order heat conduction is presented:

$$H(x) = g \quad (15)$$

where g is the heat flow (a constant) of the material.

By submitting eq. (13) into eq. (14), and taking the Laplace transform, it results:

$$\frac{\Re(\nu)s}{\nu + (1-\nu)s} T(s) = -\frac{g}{K} \quad (16)$$

which leads to:

$$T(s) = \frac{-g[\nu + (1-\nu)s]}{K\Re(\nu)s} \quad (17)$$

Taking the inverse Laplace transform of eq. (17), we obtain:

$$T(x) = -C \left[\frac{g\nu x}{K\Re(\nu)} + \frac{g(1-\nu)}{K\Re(\nu)} \right] \quad (18)$$

where C is a constant depending on the initial value $T(x)$.

The corresponding graphs with different orders $\nu = \{0.3, 0.6, 1\}$ are shown in fig. 1.

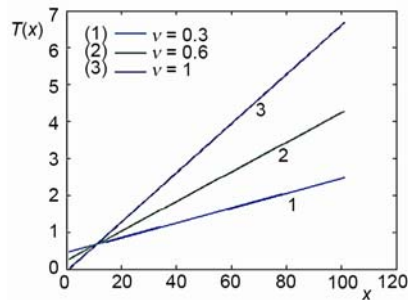


Figure 1. The plots of $T(x)$ with the parameters $\nu = \{0.3, 0.6, 1\}$, $C = -1$, $g = 2$, $K = 3$, and $\Re(\nu) = 1$

Conclusion

In this work a new fractional-order operator without singular kernel, which is an analog of the Riemann-Liouville fractional derivative with singular kernel, was proposed for the first time. An illustrative example for modelling the fractional-order steady heat flow was given and the analytical solution for the governing equation involving the fractional derivative without singular kernel was discussed.

Nomenclature

$D_0^{(\nu)}$ – fractional derivative without singular kernel, [–]	$L(\bullet)$ – Laplace transform, [–]
$H(x)$ – heat flux density, [Wm^{-2}]	$T(x)$ – temperature distribution, [K]
K – thermal conductivity, [$\text{Wm}^{-1}\text{K}^{-1}$]	x – space co-ordinate, [m]

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