# SINGULARLY PERTURBED BURGERS-HUXLEY EQUATION BY A MESHLESS METHOD

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*A meshless method based upon radial basis function (RBF) is utilized to approximate the singularly perturbed Burgers-Huxley (SPBH) equation with the viscosity coefficient $\varepsilon$. The proposed method shows that the obtained solutions are reliable and accurate. Convergence analysis of method was analyzed in a numerical way for different small values of singularity parameter.*

Key words*: Singularly perturbed Burgers-Huxley equation, Radial basis function method, Multiquadratic.*

## 1. Introduction

Nonlinear partial differential equations (NLPDEs) usually arise in modeling of various phenomena in most of the engineering and physical science branches. In the spatially homogeneous media, behaviour of bifurications and periodic trawling waves in excitable media are different. This difference cause by the strongly nonlinearity and singular characteristics of the local reaction kinetics play significant role. Indeed, singular perturbation theory utilize the mentioned characteristics of excitable media. We recall that KKP-Fisher [1] equation can be utilized sucessfully in modelling the diffucuion phenomena which admits a traveling front solution connecting the two steady states. Among possible generalizations of the Fisher equation, the Burgers-Huxley (BH) equation is most important one of the form:

$$-\frac{\partial^2 u}{\partial x^2}+\alpha u\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}-\beta u(1-u)(u-\gamma)=0. \qquad (1)$$

It is well-known that a large class of physical phenomena such as the interaction between convection effects, reaction mechanism, and diffusion transports can be described by the BH equation. Moreover, the BH equations are sucessfully applied to describe some ecological models. Let us consider a population as breeding in a medium, then the corresponding dynamical system is

$$n_t=-kn+\kappa m(n)n^2+D\Delta n$$

where $m(n)$ is the mass of food and $n$ is the size of the population per unit volume [2]. When $\alpha=0$, the BH equation can be assumed as (1) when the mass of food varies as $m(n)=m_0\left(1-\frac{n}{n_0}\right)$. More appropriate parabolic type of Eq. (1) which characterized by the singular perturbation parameter, is SPBH equation. Actually, many standard numerical approaches are not converges for these types of equations and there are only few type of numerical methods that are successful for these problems. The SPBH equation with the initial and boundary conditions is as follows:

$$-\varepsilon\frac{\partial^2 u}{\partial x^2}+\alpha u\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}-\beta u(1-u)(u-\gamma)=0, \tag{2}$$

$$(x,t)\in D\equiv\Omega_X\times\Omega_t\equiv(0,1)\times(0,T], \tag{3}$$

$$u(x,0)=u_0(x),\ \ x\in\overline{\Omega}_x, \tag{4}$$

$$u(0,t)=s_0(t),\ \ u(1,t)=s_1(t),\ \ t\in\overline{\Omega}_t, \tag{5}$$

where $\alpha$, $\beta$ and $\gamma$ parameters that take the values $\alpha\geq 0$, $\beta\geq 0$ and $\gamma\in(0,1)$. Here $\varepsilon\in(0,1]$ denotes the singular perturbation parameter. When $\varepsilon\to 0$, an outflow boundary layer is turned out in the neighborhood of right part of the lateral surface of the domain and the model can be investigated as a non-linear singularly perturbed problem. For the case $\alpha>0$, the solution of SPBH equation (2) with small value of singular perturbation parametere $\varepsilon$, demonstrates an exponential outflow boundary layer in $\Gamma_r=\{(x,t):x=1,t\in\overline{\Omega}_t\}$. We recall that most of the standard numerical approaches can approximate the solution of BH equation without small singular perturbation parameter ε. The cumbersome procedure arising in the numerical treatment even of linear singularly perturbed equations are well known [3,4,5].

The BH equation have been investigated by many researchers in recent years. The approximate analytical solutions of BH equation were obtained by the Homotopy Analysis Method [6]. Alos, some travelling wave solutions corresponding to the generalized BH equation are obtained within the first integral method in [7]. The variatioal iteration method is utilized in [8] to solve this equation without discretization. Liu et al. in [9] reported a class of multi-Soliton solutions for the generalized BH equation. Kinks and periodic wave solutions were build in [10] by utilizing the tanh-coth method.

Wang et al. obtained the solitary wave solution of generalized BH equation in [11] and multi-Soliton solutions of this equation have been studied by Liu et al. in [12]. Pseudospectral method and Darvishi's preconditioning are utilized to approximate this equation by Javidi [13]. Khattak [14] applied the radial basis function method to the generalized BH equation without singular perturbation parameter and Rathish Kumar et al. approximated the SPBH equation with three-step Taylor-Galerkin method in [15]. Xie and Li [16] applied the combination of MQ-RBF and high-order temporal approximation for the Burgers equation. Some recent papers in radial basis functions are listed in [17,18,19,20,21,22].

## 2 Radial basis function approximation

In recent years, many researchers have investigated physical problems with radial basis functions method. Published papers in this field and comparisons with other methods shows that RBF method is a powerful, reliable and convergent method for most of the physical and engineering problems. In this paper we apply this method to approximate a singular problem which is of parabolic type and many approaches cannot approximate this problem.

Let us to write the unknown function $u(X)$ as linear combination of $M$ radial functions as follows:

$$u(X) = \sum_{j=1}^{M} \lambda_j \phi_j(X) + \Psi(X), \quad X \in \Omega \subset R^d, \tag{6}$$

where $X = (x_1, x_2, \ldots, x_d)^T$ and $d$ is the dimension of the problem's domain. Moreover, in (6), $\lambda_j,\, j = 1, \ldots, M$ are unknown coefficients to be determined and $\phi_j$ are the radial basis functions and $\Psi$ is an polynomial which impose to the problem in order to reduce the condition number of condition number of discretized system. Two major class of radial basis functions are exist, i.e. infinitely smooth RBFs and other ones which are not infinitely smooth at centers. Multiquadric (MQ), Gaussian (GA), inverse multiquadric (IMQ), inverse quadric (IQ) are in the first class and thin plate spline, cubic, linear are in the second one. In the current paper we use the MQ functions defined by

$$\phi_j(X) = \sqrt{r_j^2 + c_j^2}, \tag{7}$$

where $r_j = \|X - X_j\|$ is the Euclidian norm and $c_j$ is the free shape parameter.

If $L_q^d$ denotes the space of at most $q$ order of d-variate polynomials, and supposing $L_q^d = span\{p_1, \ldots, p_m\}$ concludes:

$$\Psi(X) = \sum_{i=1}^{m} \xi_i p_i(X), \tag{8}$$

where $m = (q - 1 + d)!/(d!(q - 1)!)$.

So, to determine the unknown vector of coefficients $(\lambda_1, \ldots, \lambda_M)$ and $(\xi_1, \ldots, \xi_m)$, the collocation method can be used. Obviously, in addition to the $M$ equations extracted from collocating equation (6), we need to $m$ equations to determine these unknown coefficients. We impose the $m$ conditions for Eq. (6) as follows:

$$\sum_{j=1}^{M} \xi_i p_i(X_j) = 0, \; i = 1, \ldots, m. \tag{9}$$

It can be noted that, for any partial differential operator $\wp$ of linear type, we have

$$\wp u(X) = \sum_{j=1}^{M} \lambda_j \wp \phi_j(X) + \wp \Psi(X). \tag{10}$$

Imposing this equality into the original equation helps our to determine the unknown coefficients.

## 3 Numerical approximation schemes

In order to approximate the SPBH equation, a new numerical scheme based upon a compact form of second-order finite difference method for time approximation and MQ-RBF for a spatial approximation, is constructed (scheme I). Notice that this scheme will be utiliezed in the numerical examples of this paper. Another approximate method (scheme II) based upon the method of line [23, 24] is also represented in detail for comparison.

### 3.1 Scheme I

Let us to discretize the time derivative in (2) using the first-order forward difference, as follows:

$$-\varepsilon u_{xx}^n + \alpha u^n u_x^n + \frac{u^{n+1}-u^n}{\tau} - \beta u^n u^n + \beta\gamma u^n - \beta(1+\gamma)(u^n)^2 + \beta(u^n)^3 - e^n = 0, \quad n \ge 0 \tag{11}$$

where $\tau$ is the time step, $u^n = u(x, n*\tau)$ and $e^n$ is the truncation error given by

$$e^n = \frac{\tau}{2} u_{tt}^n + O(\tau^2). \tag{12}$$

Time differentiating of (2) with respect to time variable, leads to a higher order compact finite difference scheme. Therefore, we have

$$u_{tt} = \varepsilon u_{xxt} - \alpha u_t u_x - \alpha u u_{xt} + 2\beta(1+\gamma)uu_t - \beta\gamma u_t - 3\beta u_t u^2 = 0, \tag{13}$$

and time discretizing of (13), we obtain

$$u_{tt}^n = \varepsilon[\frac{u_{xx}^{n+1}-u_{xx}^n}{\tau}] - \alpha u_x^n[\frac{u^{n+1}-u^n}{\tau}] - \alpha u^n[\frac{u_x^{n+1}-u_x^n}{\tau}] + 2\beta(1+\gamma)u^n[\frac{u^{n+1}-u^n}{\tau}]$$
$$-\beta\gamma[\frac{u^{n+1}-u^n}{\tau}] - 3\beta(u^n)^2[\frac{u^{n+1}-u^n}{\tau}] + O(\tau) \tag{14}$$

Substituting (14) into (12) concludes

$$e^n = \frac{\varepsilon}{2}[u_{xx}^{n+1}-u_{xx}^n] - \frac{\alpha}{2}u_x^n[u^{n+1}-u^n] - \frac{\alpha}{2}u^n[u_x^{n+1}-u_x^n] + \beta(1+\gamma)u^n[u^{n+1}-u^n]$$
$$-\frac{\beta\gamma}{2}[u^{n+1}-u^n] - \frac{3}{2}\beta(u^n)^2[u^{n+1}-u^n] + O(\tau^2). \tag{15}$$

After substituting (15) into (11) we obtain our numerical scheme as follow:

$$[1+\frac{\alpha\tau}{2}u_x^n - \beta\tau u^n + \frac{\beta\gamma\tau}{2} + \frac{3\beta\tau}{2}(u^n)^2 - \beta\gamma\tau u^n]u^{n+1} - \frac{\varepsilon\tau}{2}u_{xx}^{n+1} + \frac{\alpha\tau}{2}u^n u_x^{n+1} =$$
$$[1-\frac{\beta\gamma\tau}{2} + \frac{\beta\tau}{2}(u^n)^2]u^n + \frac{\varepsilon\tau}{2}u_{xx}^n. \tag{16}$$

Now, at each time level $n$, we approximate $u^n$ by the modified $MQ$ method introduced by Kansa [25]:

$$u^n(x) = \sum_{j=0}^{M} \lambda_j^n \sqrt{(x-x_j)^2 + c_j^2} + \lambda_{M+1}^n x + \lambda_{M+2}, \tag{17}$$

where $x_j = \frac{j}{M}, j = 0,1,\ldots,M$. The $c_j$ are shape parameters which affect the accuracy of solutions.

In order to determine the $(M+3)$ unknown coefficients $\lambda_j^{n+1}, j = 0,1,\ldots,M+2,$ in the $(n+1)^{th}$ time level, firstly two boundary conditions (11) are used as follows:

$$u^{n+1}(x_0) = s_0(t), u^{n+1}(x_1) = s_1(t), \tag{18}$$

and then $(M+1)$ distinct points $\hat{x}_j = \frac{j}{M+2}$ in (0,1) using (16).

### 3.2 Scheme II

In this approximation method, spatial derivatives are firstly approximated using the RBF method, and the governing equation is reduced to a system of nonlinear ODEs. Then, the resulting system of ODEs have been solved using the fourth-order Runge-Kutta (RK4). Moreover, a function $u(x,t)$ can be approximated by

$$u^M(x,t) = \sum_{j=1}^{M} \lambda_j \phi_j(x) = \Phi^T(x)\lambda, \tag{19}$$

where $M$ is the total number of distinct points $x_j$, $j = 1,2,\ldots,M$ in [a,b], and

$$\Phi(x) = [\phi_1(x),\phi_2(x),...,\phi_M(x)]^T, \quad \lambda = [\lambda_1,\lambda_2,...,\lambda_M]^T. \tag{20}$$

Let $u^M(x_i,t) = u_i(t) = u(x_i,t)$, then (19) becomes

$$A\lambda = U, \tag{21}$$

where $U = [u_1(t),u_2(t),...,u_M(t)]^T$, and the coefficient matrix:

$$A = \begin{bmatrix} \Phi^T(x_1) \\ \Phi^T(x_2) \\ \vdots \\ \Phi^T(x_M) \end{bmatrix} = \begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_M(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_M(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_M) & \phi_2(x_M) & \cdots & \phi_M(x_M) \end{bmatrix}.$$

From (19) and (21) we can write

$$u^M(x,t) = \Phi^T(x)A^{-1}U = \Lambda(x)U, \tag{22}$$

where $\Lambda(x) = \Phi^T(x)A^{-1} = [\Lambda_1(x),\Lambda_2(x),...,\Lambda_M(x)]$. From the RBF approximation we obtain

$$u(x,t) \approx u^M(x,t) = \sum_{j=1}^{M}\lambda_j\phi_j(x) = \Phi^T(x)\lambda = \Phi^T(x)A^{-1}U = \Lambda(x)U. \tag{23}$$

Applying (23) to (2), and collocating at the node $x_i$, we have

$$\frac{du_i}{dt} - \varepsilon\frac{\partial^2\Lambda}{\partial x^2}(x_i) + \alpha u_i\frac{\partial\Lambda}{\partial x}(x_i) - \beta u_i(1-u_i)(u_i-\gamma) = 0, \quad i = 1,2,\ldots,M. \tag{24}$$

This system of equations can be written in the compact form:

$$\frac{dU}{dt} - \varepsilon(\Lambda_{xx}U) + \alpha U*(\Lambda_x U) + \beta\gamma U - \beta(1+\gamma)U^2 + \beta U^3 = 0, \tag{25}$$

where "*" denotes the two vectors component-by-component multiplication. Eq. (25) can also be written as

$$\frac{dU}{dt} = \Xi(U), \tag{26}$$

where

$$\Xi(U) = \varepsilon(\Lambda_{xx}U) - \alpha U*(\Lambda_x U) - \beta\gamma U + \beta(1+\gamma)U^2 - \beta U^3. \tag{27}$$

Related initial condition is $U^0 = [g_0(x_1), g_0(x_2),\cdots, g_0(x_M)]^T$, and from (5), we can write

$$u_1(t) = f_1(t), \quad u_M(t) = f_2(t). \tag{28}$$

Eq. (26) can be solved using the fourth order Runge-Kutta $(RK4)$. Also, solving this system of nonlinear ODEs, concludes the unknown coefficients $\lambda$ by using the relationship shown in Eq. (21).

## 4 Numerical examples

Below we present two examples with different initial and boundary conditions to illustrate the power and convergence of two explained schemes in the previous section.

### 4.1 Example 1

Below we discuss the following BH equation without the singularity parameter:

$$\frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} - \beta u(1-u)(u-\gamma) = 0, \tag{29}$$

with conditions

$$u(x,0) = \frac{\gamma}{2} + \frac{\gamma}{2} tanh(A_1 x),\ x \in \overline{\Omega}_x \tag{30}$$

and

$$u(0,t) = \frac{\gamma}{2} + \frac{\gamma}{2} tanh(-A_1 A_2 t),\ u(1,t) = \frac{\gamma}{2} + \frac{\gamma}{2} tanh(A_1(1 - A_2 t)), t \in \overline{\Omega}_t, \tag{31}$$

The corresponding exact solitary wave solution [26,27] is given by

$$u(x,t) := \frac{\gamma}{2} + \frac{\gamma}{2} tanh(A_1(x - A_2 t)) \tag{32}$$

where

$$A_1 = \frac{-\alpha + \sqrt{(\alpha^2 + 8\beta)}}{8}, A_2 = \frac{\gamma\alpha}{2} - \frac{(2-\gamma)(-\alpha + \sqrt{(\alpha^2 + 8\beta)}}{4}. \tag{33}$$

Numerical results with the parameters $\alpha = 0.5$, $\beta = 1$ and $\gamma = 0.001$ are reported in Table 1. It can be seen from this table that results of scheme I are more satisfactory than the second one.

**Table 1.** Comparison of numerical and exact solutions with $M = 10$, $T = 1$ and $\tau = 0.1$ for example 1 with uniform mesh at various mesh points.

| $x_i$ | $Exact\ solution$ | $S - I$ | $S - II$ |
|---|---|---|---|
| 0.1 | $6.370e-004$ | $6.355e-004$ | $5.148e-004$ |
| 0.2 | $6.506e-004$ | $6.436e-004$ | $5.296e-004$ |
| 0.3 | $6.640e-004$ | $6.530e-004$ | $5.444e-004$ |
| 0.4 | $6.771e-004$ | $6.637e-004$ | $5.590e-004$ |
| 0.5 | $6.889e-004$ | $6.757e-004$ | $5.736e-004$ |
| 0.6 | $7.024e-004$ | $6.890e-004$ | $5.880e-004$ |
| 0.7 | $7.147e-004$ | $7.035e-004$ | $6.023e-004$ |
| 0.8 | $7.266e-004$ | $7.192e-004$ | $6.164e-004$ |
| 0.9 | $7.382e-004$ | $7.360e-004$ | $6.301e-004$ |

### 4.2 Example 2

This example is an SPBH equation defined by:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \varepsilon\frac{\partial^2 u}{\partial x^2} = (1-u)(u-0.5)u, \tag{34}$$

with sinusoidal initial condition

$$u(x,0) = sin(\pi x),\quad x \in \overline{\Omega}_x \tag{35}$$

and

$$u(0,t) = 0,\ u(1,t) = 0,\ t \in \overline{\Omega}_t. \tag{36}$$

In this example we utilize the major method of this paper (Scheme I) to approximate the solution of problem. Let us to take $M = 8$, $T = 1$ and $\tau = 0.1$ and we multiply $M$ by $2$. The exact solution of SPBH equation for small values of the viscosity coefficient $\varepsilon$, is not available. Hence, to show the performance of this scheme at low singular perturbation parameter $\varepsilon$, we use to estimate the pointwise error as follows:

$$e_{\varepsilon}^{M,\tau} = | u^{M}(x_i, t^n) - u^{2M}(x_i, t^n) | . \qquad (37)$$

Moreover, the maximum nodal error for each $\varepsilon$, have the following form:

$$E_{\varepsilon}^{M,\tau} = \max_{i,n} e_{\varepsilon}^{M,\tau} . \qquad (38)$$

In Table 2, maximum nodal errors for different values of $\varepsilon$ up to $2^{-8}$, and different knots have been demonstrated. Indeed, this table shows that maximum nodal error decreases when point numbers incease which this fact demonstrate the numerical stability of current method. Also, Table 3 shows the comparison of obained results with current method and monotone finite difference scheme [28] for different values of the viscosity coefficient. Results of our method are more beter than the reported results in reference [28], specially for small values of singularity parameter. Finally, figures 1-4 demonstrate approximate solutions of SPBH equation in example 2 with $\varepsilon = 2^{0}, 2^{-6}, 2^{-12}$ and $2^{-24}$, respectively. Figures of approximate solutions in $x$ direction and different time values are ploted beside of each approximate solution in three dimension to show the behave of singularity parameter in the problem.

**Table 2:** Maximum nodal errors with $T = 1$ and $\tau = 0.1$ for Example 2 with uniform mesh at various singular perturbation parameters.

| $\varepsilon$ \ $M$ | 4 | 6 | 8 |
|---|---|---|---|
| $2^{0}$ | $1.901e-05$ | $3.767e-06$ | $1.017e-07$ |
| $2^{-2}$ | $2.360e-04$ | $1.110e-04$ | $3.124e-05$ |
| $2^{-4}$ | $1.370e-02$ | $1.710e-03$ | $1.22e-04$ |
| $2^{-6}$ | $2.098e-01$ | $1.987e-02$ | $8.081e-03$ |
| $2^{-8}$ | $5.474e-01$ | $2.294e-01$ | $1.725e-02$ |

**Table 3:** Comparison of maximum pointwise errors for Example 2 with the parameters $\alpha = 1$, $\beta = 1$ and $\gamma = 0.5$ on uniform mesh.

| $\varepsilon$ | $M = 16$ (Our method) | $M = 16$ (Reference [28]) | $M = 32$ (Our method) | $M = 32$ (Reference [28]) |
|---|---|---|---|---|
| $2^{0}$ | $5.3220e-08$ | $6.8456e-06$ | $1.6764e-09$ | $4.0455e-07$ |
| $2^{-2}$ | $4.7534e-06$ | $1.0777e-03$ | $5.7923e-07$ | $5.8450e-04$ |
| $2^{-4}$ | $3.5695e-05$ | $5.4069e-03$ | $2.5473e-06$ | $1.3862e-03$ |
| $2^{-6}$ | $6.6293e-04$ | $3.7340e-01$ | $3.0004e-05$ | $7.3680e-02$ |
| $2^{-8}$ | $2.3130e-03$ | 1.3865 | $9.3234e-04$ | 1.1618 |
| $2^{-10}$ | $1.9313e-02$ | 1.6736 | $8.3640e-03$ | 1.9856 |

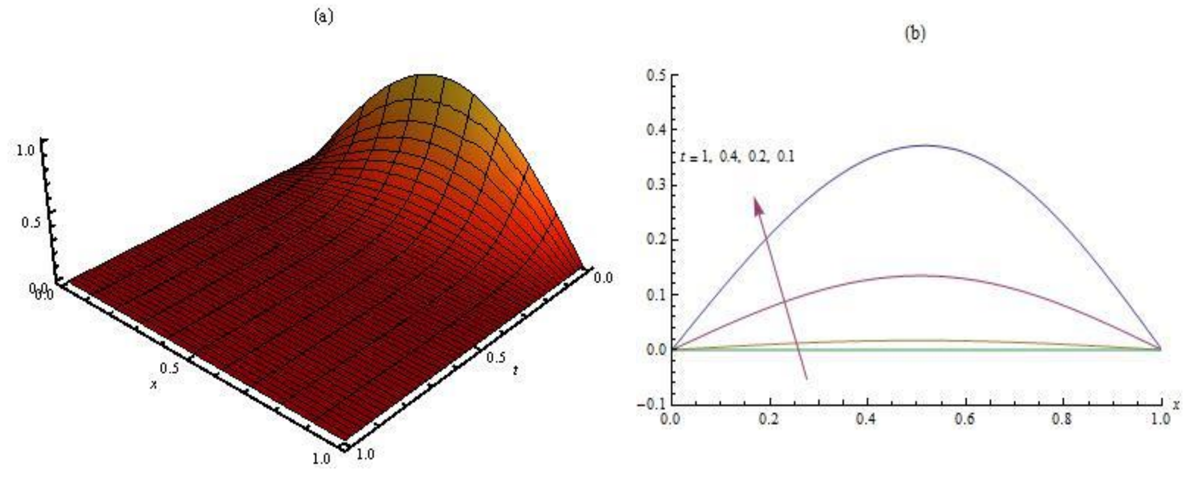


Figure 1: Approximate profile of Example 2 with $M = 10$ and $\tau = 0.01$ and $\varepsilon = 2^{0}$

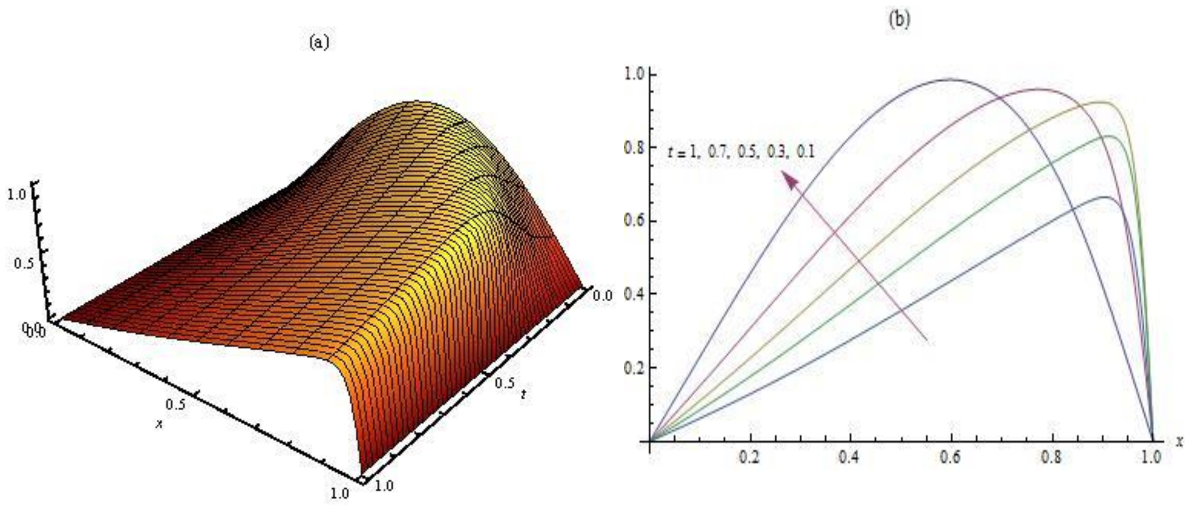


Figure 2: Approximate profile of Example 2 with $M = 10$ and $\tau = 0.01$ and $\varepsilon = 2^{-6}$

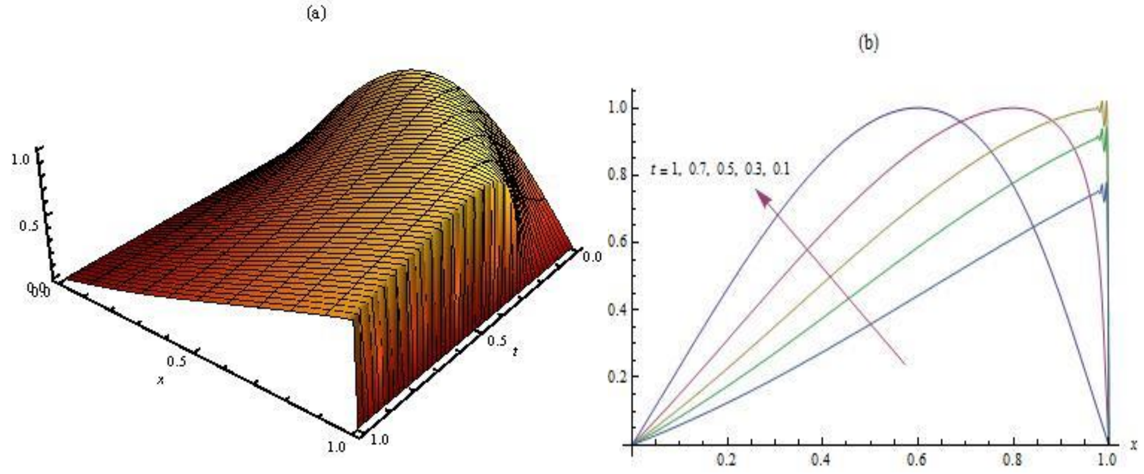


Figure 3: Approximate profile of Example 2 with $M = 10$ and $\tau = 0.01$ and $\varepsilon = 2^{-12}$

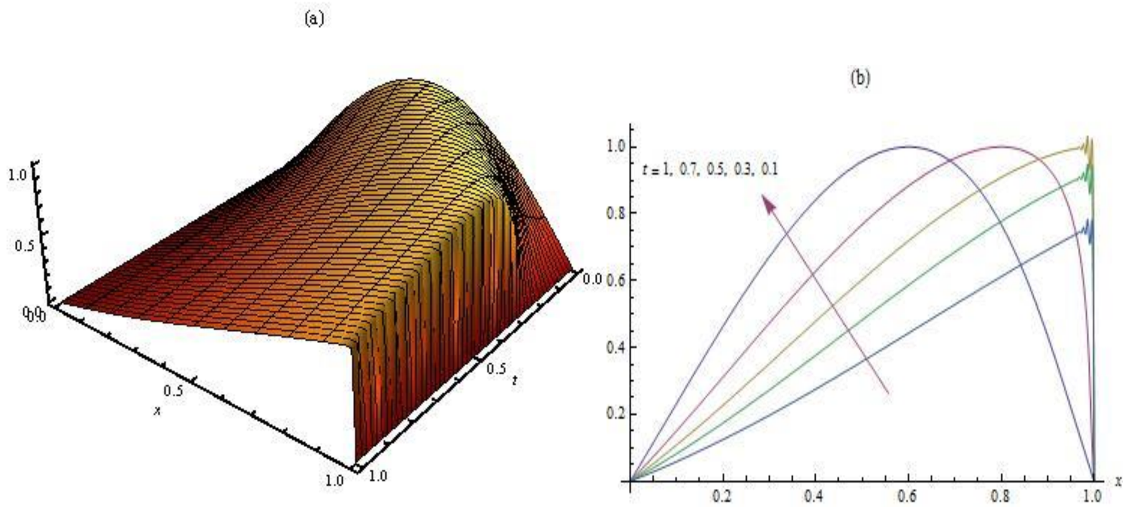


Figure 4: Approximate profile of Example 2 with $M = 10$ and $\tau = 0.01$ and $\varepsilon = 2^{-24}$

## 5 Conclusions

In this manuscript we suggest a powerful and efficient method to solve the SPBH equation using a second-order compact finite difference scheme for time discretization and MQ-RBF for spatial approximation. Convergence of the proposed method was demonstrated for different values of singularity parameter. High accuracy and efficiency of the method were demonstrated by data presented in our tables and figures.

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