ON LOCAL FRACTIONAL VOLTERRA INTEGRAL EQUATIONS IN FRACTAL HEAT TRANSFER

by

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In the article, the fractal heat-transfer models are described by the local fractional integral equations. The local fractional linear and non-linear Volterra integral equations are employed to present the heat transfer problems in fractal media. The local fractional integral equations are derived from the Fourier law in fractal media.

Key words: *fractal heat transfer, Fourier law, local fractional calculus, integral equations*

Introduction

Integral equations have played important roles in engineering applications [1]. One of important applications is to deal with the heat flux in inverse heat conduction [2-6], and so on.

Local fractional calculus [7-11], has used to describe the non-differentiable problems in heat transfer. For example, the heat-conduction equation in fractal vector space was proposed in [12-14]. Many numerical and analytical technologies for fractal heat-transfer problems were also developed, such as Sumudu transform series expansion method [15], Laplace transform series expansion method [16], variational iteration algorithm [17, 18], Laplace transform variational iteration method [19], decomposition method [20, 21], Laplace transform decomposition method [22], and homotopy perturbation method [23, 24] via local fractional operator.

The main aim of this article is to propose the fractal heat-transfer models described by the local fractional integral equations.

Fourier law of heat conduction in fractal media

The fractal temperature field reads as [12, 13, 18]:

$$\Lambda(x, y, z, \tau) = f(x, y, z, \tau) \text{ at } \tau > \tau_0 \text{ and in } \Omega_\alpha \tag{1}$$

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where $f(x,y,z,\tau)$ is a non-differentiable function in fractal domain Ω_α.

For a given fractal temperature field $\Lambda(x,y,z,\tau)$, a local fractional temperature gradient of $\Lambda(x,y,z,\tau)$ can be written [12-14, 18]:

$$\nabla^\alpha \Lambda(x,y,z,\tau) = \frac{\partial^\alpha \Lambda(x,y,z,\tau)}{\partial x^\alpha}\vec{e}_1^{\,\alpha} + \frac{\partial^\alpha \Lambda(x,y,z,\tau)}{\partial y^\alpha}\vec{e}_2^{\,\alpha} + \frac{\partial^\alpha \Lambda(x,y,z,\tau)}{\partial z^\alpha}\vec{e}_3^{\,\alpha} \tag{2}$$

where ∇^α is the local fractional derivative operator in fractal time-space [12-14].

We consider the fractal heat flux per unit fractal area, denoted by $\vec{q}(x,y,z,t)$, is proportional to the fractal temperature gradient in fractal media. Fourier law of heat conduction in fractal media can be written [12-14]:

$$\vec{q}(x,y,z,t) = -\mu \nabla^\alpha \Lambda(x,y,z,t) \tag{3}$$

where μ is the thermal conductivity of the fractal material, which is related to the fractal dimensions of materials. Meanwhile, we observe that $\vec{q}(x,y,z,t)$ is the fractal Fourier flow and that $\Lambda(x,y,z,t)$ is the non-differentiable temperature field.

Fourier law of low-dimensional heat conduction equation in fractal media is given by:

$$q(x,t) = -\mu \frac{\partial^\alpha \Lambda(x,t)}{\partial x^\alpha}, \quad \text{at } \tau > \tau_0 \quad \text{and in } \Gamma_\alpha \tag{4}$$

where μ is the thermal conductivity of the fractal material and $\partial^\alpha/\partial x^\alpha$ is the local fractional partial derivative operator with respect to the space x [7, 12, 13].

When $t = t_0$, eq. (4) can be written:

$$q(x) = -\mu \frac{\mathrm{d}^\alpha \Lambda(x)}{\mathrm{d}x^\alpha} \tag{5}$$

where μ is the thermal conductivity of the fractal material and $\mathrm{d}^\alpha/\mathrm{d}x^\alpha$ is the local fractional derivative operator [7-25]. Equtation (5) can be applied to describe the fractal steady heat conduction problems.

The Volterra integral equation in fractal heat transfer

In order to obtain the Volterra integral equations in fractal heat transfer, eq. (5) can be written:

$$\frac{\mathrm{d}^\alpha \Lambda(x)}{\mathrm{d}x^\alpha} = q[x,\Lambda(x)], \quad (0 \le x \le l) \tag{6}$$

where $q(x) = -\mu q[x,\Lambda(x)]$.

With the help of the result [25], we have the local fractional Voltterra integral equation of second kind:

$$\Lambda(x) = \Lambda(0) + \frac{1}{\Gamma(1+\alpha)}\int_0^x q[\tau,\Lambda(\tau)](\mathrm{d}\tau)^\alpha \tag{7}$$

where $\int_0^x (\mathrm{d}\tau)^\alpha / \Gamma(1+\alpha)$ is a local fractional integral operator [7, 12, 13, 25].

When the fractal flow is decomposed into the linear term of non-differentiable type:

$$q[x, \Lambda(x)] = \beta_0(x) + \beta_1(x)\Lambda(x) \tag{8}$$

Equation (7) is rewritten in the form:

$$\Lambda(x) = \Lambda(0) + \frac{1}{\Gamma(1+\alpha)} \int_0^x [\beta_0(\tau) + \beta_1(\tau)\Lambda(\tau)](\mathrm{d}\tau)^\alpha \tag{9}$$

where $\beta_0(x)$ and $\beta_1(x)$ are two parameters related to the fractal heat flow.

Taking:

$$\phi(x) = \Lambda(0) + \frac{1}{\Gamma(1+\alpha)} \int_0^x \beta_0(\tau)(\mathrm{d}\tau)^\alpha \tag{10}$$

we can rewrite eq. (9):

$$\Lambda(x) = \phi(x) + \frac{1}{\Gamma(1+\alpha)} \int_0^x \beta_1(\tau)\Lambda(\tau)(\mathrm{d}\tau)^\alpha \tag{11}$$

When $\Lambda(0) = 0$ and $\beta_0(x) = 0$, we obtain the local fractional linear Voltterra integral equation of first kind:

$$\Lambda(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^x \beta_1(\tau)\Lambda(\tau)(\mathrm{d}\tau)^\alpha \tag{12}$$

Taking:

$$\beta_1(\tau) = \frac{(x-\tau)^\alpha}{\Gamma(1+\alpha)} \tag{13}$$

from eqs. (11) and (12) we get the local fractional linear Voltterra integral equations of convolution-type:

$$\Lambda(x) = \phi(x) + \frac{1}{\Gamma(1+\alpha)} \int_0^x \frac{(x-\tau)^\alpha}{\Gamma(1+\alpha)} \Lambda(\tau)(\mathrm{d}\tau)^\alpha \tag{14}$$

and

$$\Lambda(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^x \frac{(x-\tau)^\alpha}{\Gamma(1+\alpha)} \Lambda(\tau)(\mathrm{d}\tau)^\alpha \tag{15}$$

When:

$$\beta_1(\tau) = 1 \tag{16}$$

the local fractional linear Voltterra integral equations of convolution-type are:

$$\Lambda(x)=\phi(x)+\frac{1}{\Gamma(1+\alpha)}\int_0^x \Lambda(\tau)(\mathrm{d}\tau)^{\alpha} \tag{17}$$

and

$$\Lambda(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^x \Lambda(\tau)(\mathrm{d}\tau)^{\alpha} \tag{18}$$

When the fractal flow is decomposed into the non-linear term of non-differentiable type:

$$q[x,\Lambda(x)]=\chi(x)\Lambda^2(x) \tag{19}$$

we obtain the local fractional non-linear Volterra integral equation of second kind:

$$\Lambda(x)=\Lambda(0)+\frac{1}{\Gamma(1+\alpha)}\int_0^x \chi_3(x)\Lambda^2(x)(\mathrm{d}\tau)^{\alpha} \tag{20}$$

where $\chi(x)$ is the parameter related to the fractal heat flow.

In view of eq. (19), the local fractional non-linear Volterra integral equation of first kind is written as:

$$\Lambda(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^x \chi(x)\Lambda^2(x)(\mathrm{d}\tau)^{\alpha} \tag{21}$$

Taking:

$$\chi(x)=\frac{(x-\tau)^{\alpha}}{\Gamma(1+\alpha)} \tag{22}$$

the local fractional non-linear Voltterra integral equations of convolution-type are:

$$\Lambda(x)=\Lambda(0)+\frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(x-\tau)^{\alpha}}{\Gamma(1+\alpha)}\Lambda^2(x)(\mathrm{d}\tau)^{\alpha} \tag{23}$$

and

$$\Lambda(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(x-\tau)^{\alpha}}{\Gamma(1+\alpha)}\Lambda^2(x)(\mathrm{d}\tau)^{\alpha} \tag{24}$$

When

$$\chi(x)=1 \tag{25}$$

we obtain the local fractional non-linear Volterra integral equations:

$$\Lambda(x)=\Lambda(0)+\frac{1}{\Gamma(1+\alpha)}\int_0^x \Lambda^2(x)(\mathrm{d}\tau)^{\alpha} \tag{26}$$

and

$$\Lambda(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^x \Lambda^2(x)(\mathrm{d}\tau)^\alpha \tag{27}$$

Conclusion

In our work we proposed the local fractional linear and non-linear Volterra integral equations in fractal heat transfer for the first time. We first consider the fractal flow is decomposed to the linear and non-linear terms of non-differentiable type. Fourier law of the heat conduction in fractal media are written into the linear and non-linear forms. The linear and non-linear steady heat conductions are transferred into the local fractional linear and non-linear Volterra integral equations. The proposed integral equations of Volterra type are efficient to describe non-differentiable behaviors of the steady heat conductions in fractal media.

Nomenclature

x, y, z – space co-ordinates, [m]
$\Lambda(x, y, z, \tau)$ – fractal temperature field, [K]
t – time, [s]
μ – specific heat of the material, [Jkg^{-1}K^{-1}]

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