HE'S FRACTIONAL DERIVATIVE AND ITS APPLICATION FOR FRACTIONAL FORNBERG-WHITHAM EQUATION

by

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Fractional Fornberg-Whitham equation with He's fractional derivative is studied in a fractal process. The fractional complex transform is adopted to convert the studied fractional equation into a differential equation, and He's homotopy perturbation method (HPM) is used to solve the equation.

Key words: *fractal derivative, Homotopy perturbation method, Fractional complex transform, fractional Fornberg-Whitham equation*

1. Introduction

In this paper, we consider the following fractional Fornberg-Whitham equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^{2+\alpha} u}{\partial x^2 \partial t^\alpha} + \frac{\partial u}{\partial x} = u\frac{\partial^3 u}{\partial x^3} - u\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

with the following initial condition:

$$u(x,0) = e^{\frac{1}{2}x}, \tag{1.2}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is He's fractional derivative defined as[1,2,3]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1} [u_0(s) - u(s)] ds, \tag{1.3}$$

where $u_0(x,t)$ is the solution of its continuous partner of the problem with the same initial condition of the fractal partner.

When $\alpha = 1$ eq.(1.1) turns out to be the original Fornberg-Whitham equation. Eq.(1.1) describes Fornberg-Whitham nonlinear wave in fractal time domain. When time tends to infinite small, time becomes discontinuous, and He's fractional derivative can describe the motion.

In the past three decades, the fractional derivatives have gained a lot of attention of physicists, mathematicians and engineers. Many kinds of interdisciplinary problems can be modeled with the help of fractional derivatives[3,4,5] in many fields of science and engineering. However, it is very difficult for us to find the exact solutions of fractional differential equations, so the analytical and approximation techniques have to be used. Many methods have been used to solve linear and nonlinear fractional differential equations. Some of recent powerful analytical methods contain the Adomain decomposition method (ADM), the variational iteration method (VIM) [6, 7, 8, 9, 10,11,12] , Exp-function Method [13, 14] and sub-equation method[15].

In this paper, we will apply He's homotopy perturbation method (HPM) [16, 17, 18,19,20] and fractional complex transform [21,22,23,24,25] to solve the fractional Fornberg-Whitham equation. The HPM is a powerful technology for finding the approximate analytical solution of linear and nonlinear problem. The method was first proposed by He [16, 17, 18, 19,20] and was successfully used to solve nonlinear problem. The fractional complex transform was first proposed by He [21,22,23,24]. The fractional complex transform is the simplest approach[25], fractional equations adopts generally discontinuous solutions, and the fractional complex transform gives a continuous solution when the scale tends to a nonzero value. The fractional complex transform can convert fractional differential equation into its differential partner, therefore the HPM can be effectively applied when we combined the fractional complex transform.

2. The He's homotopy perturbation method (HPM)

Consider the following differential equation:

$$A(u)-f(r)=0,\quad r\in\Omega, \tag{2.1}$$

with the boundary condition of:

$$B(u,\frac{\partial u}{n})=0,\quad r\in\Gamma, \tag{2.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function , Γ is the boundary of the domain Ω.

We can divide operator A into N and L, where N is a nonlinear and L is a linear operator.

Therefore eq.(2.1) can be written into the following form:

$$L(u)+N(u)-f(r)=0. \tag{2.3}$$

According to the homotopy technique, we can construct a homotopy as $\mu(r,q):\Omega\times[0,1]\to R$ which satisfies:

$$H(\mu,q)=(1-q)[L(\mu)-L(u_0)]+q[A(\mu)-f(r)]=0, \tag{2.4}$$

or

$$H(\mu,q)=L(\mu)-L(u_0)+qL(u_0)+q[N(\mu)-f(r)]=0, \tag{2.5}$$

where $q \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of eq.(2.1), which satisfies the boundary conditions. Using eq .(2.4) and eq.(2.5), we can obtain:

$$H(\mu,0) = L(\mu) - L(u_0) = 0, \tag{2.6}$$

$$H(\mu,1) = A(\mu) - f(r) = 0. \tag{2.7}$$

The changing process of q from zero to unity is just that of $\mu(r,q)$ from $u_0(r)$ to $u(r)$. This is called deformation in topology. The $L(\mu) - L(u_0)$ and $A(\mu) - f(r)$ are called homotopy. Using the HPM, we can first apply the embedding parameter q as a small parameter and assume that the solution of eq.(2.4) and eq.(2.5) can be written into a power series in term of q:

$$\mu = \mu_0 + q\mu_1 + q^2\mu_2 + q^3\mu_3 + q^4\mu_4 + \cdots. \tag{2.8}$$

Setting $q = 1$ in (2.8), we obtain:

$$u = \lim_{q\to 1} \mu = \mu_0 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \cdots. \tag{2.9}$$

The combination of homotopy method and perturbation method is called homotopy perturbation method (HPM). The HPM eliminates the drawbacks of the traditional perturbation methods . This method have full advantages of the traditional perturbation methods. The series (2.9) is convergent for most cases. However, the convergent rate depend on the nonlinear operator $N(\mu)$. Moreover, the following suggestions is given by He[16]:

(1) The second derivative of $N(\mu)$ with respect to μ must be small because the parameter may be relatively large; that is, $q \to 1$.

(2) The norm of $L^{-1}(\frac{\partial N}{\partial \mu})$ must be smaller than one so that the series converges.

3. Numerical application

The first step to solve eq.(1.1) by Homotopy perturbation method is to convert the equation into its differential partner by the fractional complex transform[21,22,23]:

$$T = \frac{t^\alpha}{\Gamma(1+\alpha)}. \tag{3.1}$$

We can easily convert eq.(1.1) into a differential equation, which is the following form:

$$\frac{\partial u}{\partial T} - \frac{\partial^3 u}{\partial x^2 \partial T} + \frac{\partial u}{\partial x} = u\frac{\partial^3 u}{\partial x^3} - u\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}, \tag{3.2}$$

with the initial condition:

$$u(x,0)=e^{\frac{1}{2}x}. \quad (3.3)$$

According to the HPM, we construct the following homotopy for eq.(3.2):

$$(1-q)u_T+q(u_T-u_{xxT}+u_x-uu_{xxx}+uu_x-3u_xu_{xx})=0. \quad (3.4)$$

Therefore, the following results are obtained:

$$q^0:\frac{\partial u_0}{\partial T}=0, \quad (3.5)$$

$$q^1:\frac{\partial u_1}{\partial T}+\frac{\partial u_0}{\partial x}-\frac{\partial^3 u_0}{\partial x^2\partial T}-u_0\frac{\partial^3 u_0}{\partial x^3}+u_0\frac{\partial u_0}{\partial x}-3\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}=0, \quad (3.6)$$

$$q^2:\frac{\partial u_2}{\partial T}+\frac{\partial u_1}{\partial x}-\frac{\partial^3 u_1}{\partial x^2\partial T}-u_0\frac{\partial^3 u_1}{\partial x^3}+u_1\frac{\partial^3 u_0}{\partial x^3}+u_0\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_0}{\partial x}$$

$$-3\frac{\partial u_0}{\partial x}\frac{\partial^2 u_1}{\partial x^2}-3\frac{\partial u_1}{\partial x}\frac{\partial^2 u_0}{\partial x^2}=0, \quad (3.7)$$

$$q^3:\frac{\partial u_3}{\partial T}+\frac{\partial u_2}{\partial x}-\frac{\partial^3 u_2}{\partial x^2\partial T}-u_0\frac{\partial^3 u_2}{\partial x^3}-u_1\frac{\partial^3 u_1}{\partial x^3}-u_2\frac{\partial^3 u_0}{\partial x^3}+u_0\frac{\partial u_2}{\partial x}+u_1\frac{\partial u_1}{\partial x}$$

$$+u_2\frac{\partial u_0}{\partial x}-3\frac{\partial u_0}{\partial x}\frac{\partial^2 u_2}{\partial x^2}-3\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2}-3\frac{\partial u_2}{\partial x}\frac{\partial^2 u_0}{\partial x^2}=0, \quad (3.8)$$

$$q^4:\frac{\partial u_4}{\partial T}+\frac{\partial u_3}{\partial x}-\frac{\partial^3 u_3}{\partial x^2\partial T}-u_0\frac{\partial^3 u_3}{\partial x^3}-u_1\frac{\partial^3 u_2}{\partial x^3}-u_2\frac{\partial^3 u_1}{\partial x^3}-u_3\frac{\partial^3 u_0}{\partial x^3}$$

$$+u_0\frac{\partial u_3}{\partial x}+u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_1}{\partial x}+u_3\frac{\partial u_0}{\partial x}-3\frac{\partial u_0}{\partial x}\frac{\partial^2 u_3}{\partial x^2}$$

$$-3\frac{\partial u_1}{\partial x}\frac{\partial^2 u_2}{\partial x^2}-3\frac{\partial u_2}{\partial x}\frac{\partial^2 u_1}{\partial x^2}-3\frac{\partial u_3}{\partial x}\frac{\partial^2 u_0}{\partial x^2}=0. \quad (3.9)$$

We set $u_0(x,T)=e^{\frac{1}{2}x}$ as the initial approximation. Then applying the eqs.(3.5)-(3.9), we can obtain:

$$u_0(x,T)=e^{\frac{1}{2}x},$$

$$u_1(x,T)=-\frac{1}{2}e^{\frac{1}{2}x}T,$$

$$u_2(x,T)=\frac{1}{8}e^{\frac{1}{2}x}(-T+T^2),$$

$$u_3(x,T) = -\frac{1}{96}e^{\frac{1}{2}x}(3T - 6T^2 + 2T^3),$$

$$u_4(x,T) = \frac{1}{384}e^{\frac{1}{2}x}(-3T + 9T^2 - 6T^3 + T^4),$$

……..

In this manner, the rest of components can be obtained. Using the HPM, we can have the approximate solution as the following form:

$$\phi_5 = u_0(x,T) + u_1(x,T) + u_2(x,T) + u_3(x,T) + u_4(x,T)$$

$$= e^{\frac{1}{2}x}(1 - \frac{85}{128}T + \frac{27}{128}T^2 - \frac{7}{192}T^3 + \frac{1}{384}T^4).$$

Substituting (3.1) into the above results, we have:

$$u_0(x,t) = e^{\frac{1}{2}x}$$

$$u_1(x,t) = -\frac{1}{2}e^{\frac{1}{2}x}(\frac{t^\alpha}{\Gamma(1+\alpha)}),$$

$$u_2(x,t) = \frac{1}{8}e^{\frac{1}{2}x}(-\frac{t^\alpha}{\Gamma(1+\alpha)} + (\frac{t^\alpha}{\Gamma(1+\alpha)})^2),$$

$$u_3(x,t) = -\frac{1}{96}e^{\frac{1}{2}x}(3(\frac{t^\alpha}{\Gamma(1+\alpha)}) - 6(\frac{t^\alpha}{\Gamma(1+\alpha)})^2 + 2(\frac{t^\alpha}{\Gamma(1+\alpha)})^3),$$

$$u_4(x,t) = \frac{1}{384}e^{\frac{1}{2}x}(-3(\frac{t^\alpha}{\Gamma(1+\alpha)}) + 9(\frac{t^\alpha}{\Gamma(1+\alpha)})^2 - 6(\frac{t^\alpha}{\Gamma(1+\alpha)})^3 + (\frac{t^\alpha}{\Gamma(1+\alpha)})^4),$$

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So, the fifth-order approximate solution of eq.(1.1) can be written into the following form:

$$\Phi_5 = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t)$$

$$= e^{\frac{1}{2}x}(1 - \frac{85}{128}\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{27}{128}(\frac{t^\alpha}{\Gamma(\alpha+1)})^2 - \frac{7}{192}(\frac{t^\alpha}{\Gamma(\alpha+1)})^3 + \frac{1}{384}(\frac{t^\alpha}{\Gamma(\alpha+1)})^4). \quad (3.10)$$

Remark3.1. When $\alpha = 1$, the exact solution of eq.(1.1) is given by the following form:

$$u(x,t) = e^{\frac{x}{2} - \frac{2t}{3}}. \quad (3.11)$$

Remark3.2. Fig.1-fig.3 show the 5th-order approximate solutions by HPM and fractional complex transform for $\alpha = 0.6$, $\alpha = 0.8$, $\alpha = 1$ respectively. In fig.4, we draw the exact solution of eq.(1.1) for $\alpha = 1$. In tab.1, we compare the exact solution with the

5th-order approximate solutions for different values of α. By comparison, it is easy to find that the approximate solutions continuously depend on the values of time-fractional derivative.

Remark3.3. Fig.5 shows the absolute error between the exact solution and the 5-order approximate solution by the proposed method for $\alpha = 1$. In tab.2, we compare the absolute error between 5-order approximate solution with the exact solution for $\alpha = 1$ at some points. The numerical results show that the method is highly accurate. In this paper, we only apply five terms. If we apply more terms, the accuracy of the approximate solution will be greatly improved.

Table 1. Comparison between the exact solution and the 5th-order approximate solution by HPM for different values of α

		α			
x	t	0.6	0.8	1	$u_{exa}(\alpha = 1)$
0.2	0.3	0.7689581621	0.8408773684	0.9049165380	0.9048374180
0.4	0.6	0.7046737702	0.7585023555	0.8182940453	0.8187307531
0.5	0.7	0.7024063150	0.7490526220	0.8046161366	0.8051983240
0.8	0.9	0.7400206617	0.7720433300	0.8180175980	0.8187307531

Table 2. Comparison between the exact solution and the 5th-order approximate solution by HPM for $\alpha = 1$

		α=1		
x	t	Φ_5	u_{exa}	$\lvert\Phi_5 - u_{exa}\rvert$
0.3	0.2	1.016997082	1.016806330	0.000190752
0.4	0.1	1.142826165	1.142630812	0.000195353
0.8	0.5	1.068606851	1.068939106	0.000332255
0.6	0.9	0.740172931	0.740818221	0.0006452894

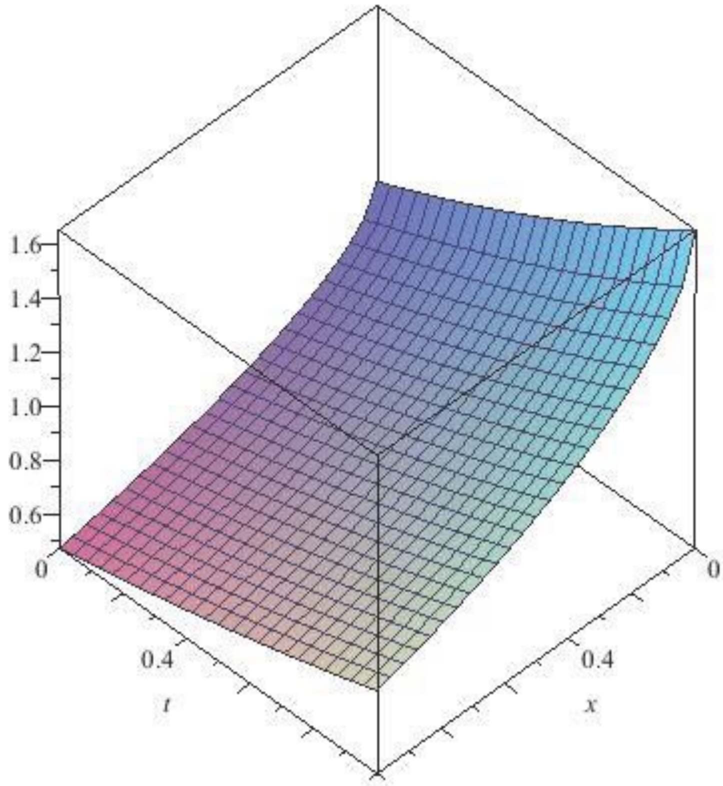

Figure. 1. The 5th-order approximate solution of eq.(1.1) for $\alpha = 0.6$ **.**

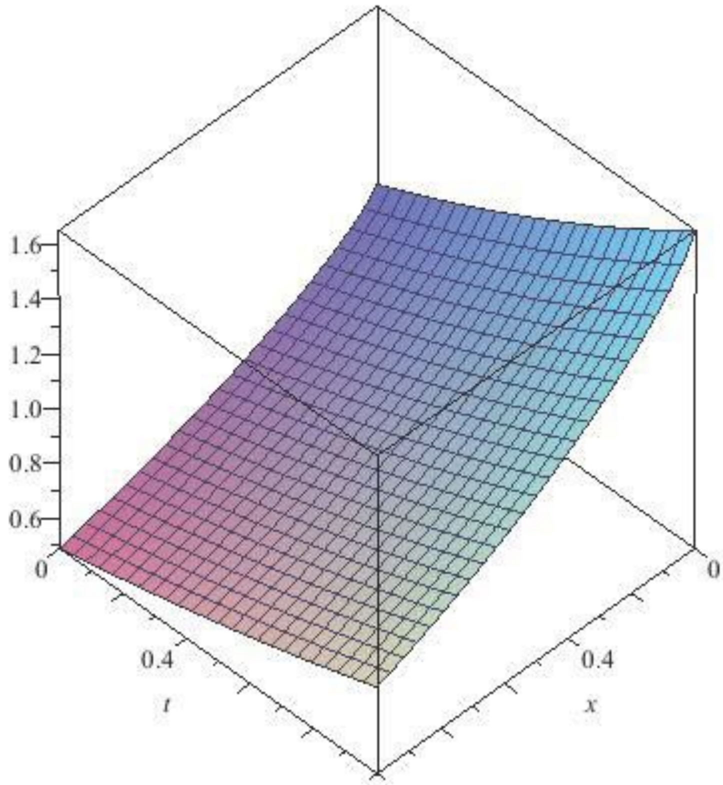

Figure. 2. The 5th-order approximate solution of eq.(1.1) for $\alpha = 0.8$**.**

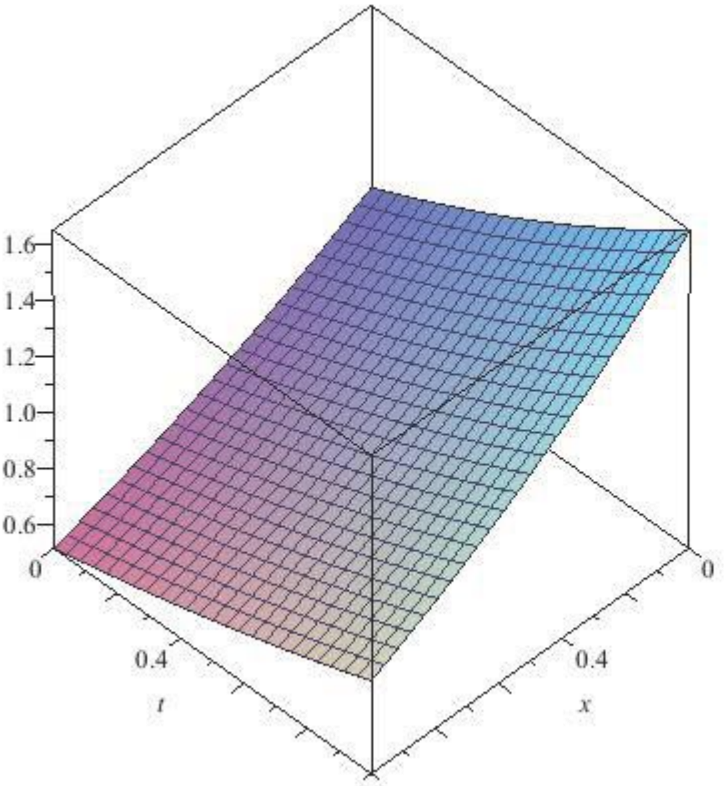

Figure. 3. The 5th-order approximate solution of eq.(1.1) for $\alpha = 1$.

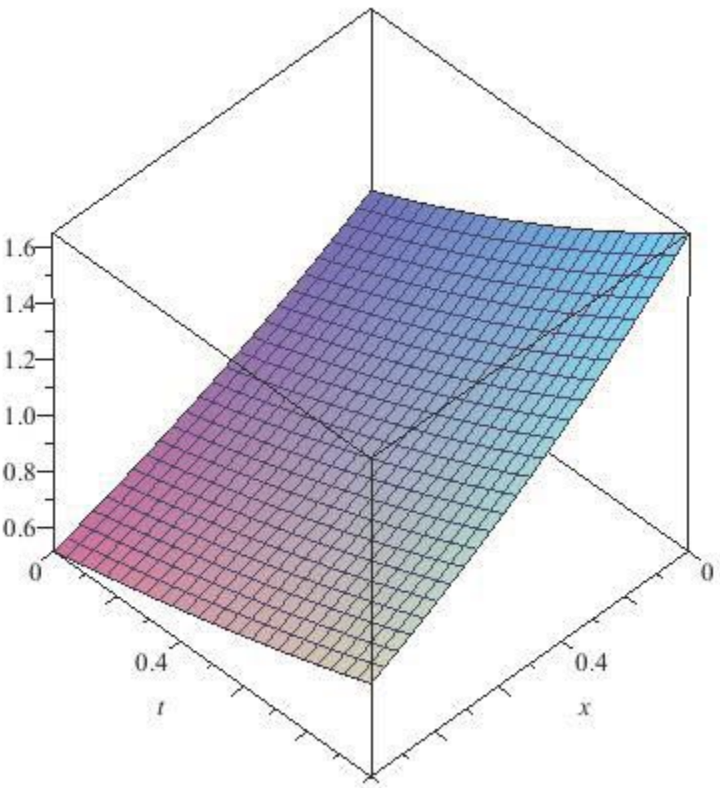

Figure. 4. The exact solution of eq.(1.1) for $\alpha = 1$.

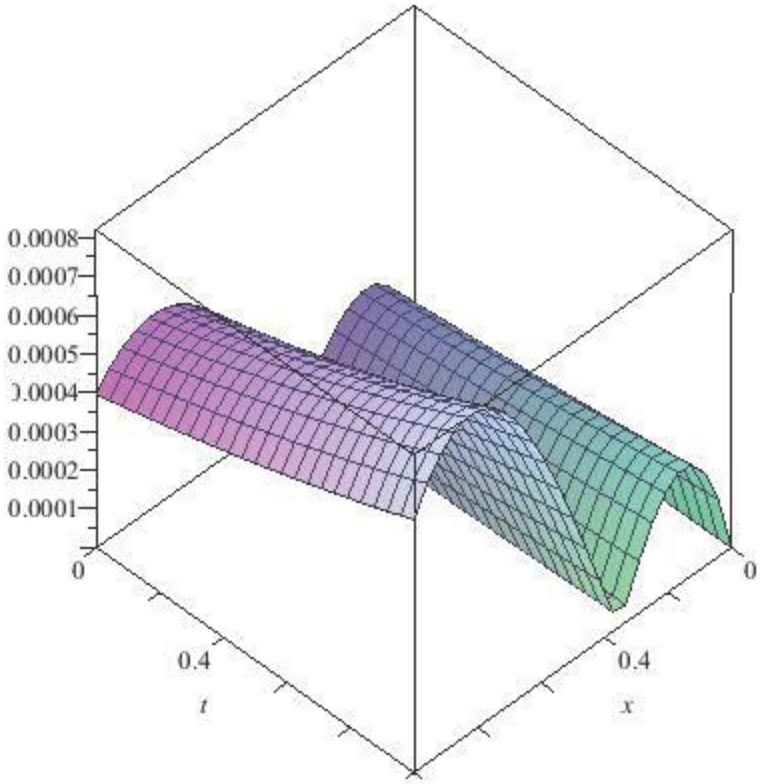

Figure. 5. The absolute error $|u_{exa} - \Phi_5|$ **for** $\alpha = 1$**.**

4. Conclusions

In this paper, based on He's fractional derivative, we combined He's homotopy perturbation method (HPM) and fractional complex transform for finding the approximate solution of the nonlinear time-fractional Fornberg-Whithan equation. The result shows that the proposed method is a very powerful, efficient and easy mathematical technology for solving the nonlinear fractional differential equations in engineering and science.

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