AN ALTERNATIVE INTEGRAL-BALANCE SOLUTION TO TRANSIENT DIFFUSION OF HEAT (MASS) BY TIME-FRACTIONAL SEMI-DERIVATIVES AND SEMI-INTEGRALS

Fixed Boundary Conditions

by

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A new approach to integral-balance solutions of the diffusion equation of heat (mass) with constant transport properties by applying time-fractional semi-derivatives and semi-integrals of Riemann-Liouville sense has been developed. The time-fractional semi-derivatives and semi-integrals replace the surface gradient (temperature) which in the classical heat-balance integral method of Goodman and the double-integration method should be expressed through the assumed profile. The application of semi-derivatives and semi-integrals reduces the approximation errors to levels less than the ones exhibited by the classical heat-balance integral method and double-integration method. The method is exemplified by solutions of Dirichlet and Neumann boundary condition problems.

Key words: *transient conduction, approximate solutions, semi-derivative, semi-integral, integral-balance method, time-fractional derivatives*

Introduction

The integral-balance method [1] to parabolic models employs a concept of a penetration depth which is physically motivated by the behaviour of the hyperbolic counterpart model of diffusion transfer, thus defining a sharp forint of propagation. Integral-balance solutions are extensively used with both pure analytical [2, 3] and practically oriented modelling studies [4-9]. The simplicity of the method is still attractive for scientist [10-12] and various modifications [13-15] have been developed.

The core of the integral method is the choice of a profile [16] which should satisfy the conditions at both ends of the thermal penetration depth. This approach is a zeroth order moment solution converting the governing partial differential equation to an ordinary differential equation with respect to the penetration depth. The common approach is to use quadratic or cubic polynomial profiles [2, 17, 18] as they originate from Goodman's classical work [1]. More flexible application of the integral balance solutions can be developed by use of a parabolic profile with unspecified exponent [10, 11, 16, 19]. The principle approach and the problems emerging in implementation of the integral-balance method are formulated next.

Transient diffusion (heat or mass) in a homogeneous medium with a constant transport coefficient (diffusivity), a, is modelled by the parabolic equation in dimensionless form (with respect to the temperature, concentration, only):

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$$\frac{\partial\theta}{\partial t}=a\frac{\partial^2\theta}{\partial x^2},\quad \theta(x,t)=0 \qquad \text{for} \quad t<0 \tag{1}$$

The simplest method known as heat-balance integral method (HBIM) [1, 20] suggests integration of eq. (1) with respect to the space co-ordinate over a finite penetration depth, δ, namely:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta \theta(x,t)\mathrm{d}x-\theta(\delta,t)\frac{\mathrm{d}\delta}{\mathrm{d}t}=\int_0^\delta a\frac{\partial^2\theta}{\partial x^2}\mathrm{d}x \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta \theta(x,t)\mathrm{d}x=-a\frac{\partial\theta}{\partial x}(0,t) \tag{2a,b}$$

In the case of a semi-infinite medium the value of $\theta(x, t)$ far away from the boundary $x = 0$ is assumed $\theta(\infty, t) = 0$. The sharp front concept replaces these conditions by the following ones:

$$\theta(\delta)=0 \quad \text{and} \quad \frac{\partial\theta}{\partial x}(\delta)=0 \tag{3a,b}$$

Replacing θ by an approximate profile θ_a, an ordinary differential equation describing the time-evolution of $\delta(t)$ can be derived. The principle problem of HBIM is the approximation of the gradient of right-side of eq. (4), *because it should be defined through the assumed profile*. A step, avoiding this problem, is the double-integration method (DIM) [11, 21] when the Dirichlet problem is at issue.

The present work addresses a new technique improving the integration step in the integral balance method. The new step considers the surface flux (the space derivative) or temperature, *to be expressed by the prescribed boundary conditions (BC) and Riemann-Liouville (RL) time-fractional semi-derivatives and integrals,* thus avoiding their expressions through the assumed profile which is a principle drawback of HBIM and DIM (when the flux is prescribed at the boundary). Two examples demonstrate how this new approach can be implemented together with the classical HBIM [1, 20] and the refined version known as DIM [11, 21].

Semi-derivative integral method (SDIM): general relationships

The relationships between the temperature and the flux at the boundary

Equation (1) can be represented as [22]:

$$\left(\frac{\partial^{1/2}\theta}{\partial t^{1/2}}-\sqrt{a}\frac{\partial\theta}{\partial x}\right)\left(\frac{\partial^{1/2}\theta}{\partial t^{1/2}}+\sqrt{a}\frac{\partial\theta}{\partial x}\right)=0,\quad \frac{\partial^{1/2}\theta}{\partial t^{1/2}}=\frac{1}{\Gamma(1/2)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^u\frac{\theta(x,t)}{\sqrt{t-u}}\mathrm{d}u-\frac{\theta(x,0)}{\sqrt{\pi t}} \tag{4a, b }$$

where $\partial^{1/2}\theta/\partial t^{1/2}=D_{RL}^{1/2}$ is the time-fractional RL derivative of order 1/2, u – the dummy variable, and $\Gamma(1/2) = \pi^{1/2}$. In eq. (4a) only the second term has a physical meaning [22]. Hence, the time-fractional equivalent of eq. (1) is eq. (5a) [22], while eq. (5b) is a relation at $x = 0$, namely:

$$\frac{\partial^{1/2}\theta}{\partial t^{1/2}}=-\sqrt{a}\frac{\partial\theta}{\partial x}\Rightarrow\frac{\partial^{1/2}\theta(0,t)}{\partial t^{1/2}}=-\sqrt{a}\frac{\partial\theta(0,t)}{\partial x} \tag{5a,b}$$

Applying the operator $D_t^{-1/2}$ to both sides of eq. (5b) we get:

$$\theta(0,t) = -\sqrt{a}\frac{\partial^{-1/2}}{\partial t^{-1/2}}\left[\frac{\partial\theta(0,t)}{\partial x}\right], \qquad \frac{\partial^{-1/2}}{\partial t^{-1/2}}\theta(x,t) = \frac{1}{\Gamma(1/2)}\frac{\sqrt{a}}{\lambda}\int_0^t \frac{\theta(x,t)}{\sqrt{t-u}}\mathrm{d}u \tag{6a,b}$$

The operator $\partial^{-1/2}\theta(x,t)/\partial t^{-1/2} = D_t^{-1/2}\theta(x,t)$ is the time-fractional (semi-integral) of order 1/2 in RL sense [22], where u is the dummy variable. In a sense, following the results of [23, 24] the heat flux and temperature at any point, can be presented explicitly by the relations:

$$q(x,t) = \frac{\lambda}{\sqrt{\alpha}}\left[\frac{\partial^{1/2}T(x,t)}{\partial t^{1/2}} - \frac{T(x,t)}{\sqrt{\pi t}}\right], \qquad T(x,t) = \frac{\sqrt{\alpha}}{\lambda}\frac{\partial^{-1/2}[q(x,t)]}{\partial t^{-1/2}} + T(x,0) \tag{7a,b}$$

which are equivalent to eqs. (5a) and (6a).

At this point we have to stress the attention on the fact that the relationships (6a), (6b), (7a), and (7b) are natural solutions in terms of the RL derivative. However, if we suggest that the RL derivative could be replaced mechanistically by the Caputo derivative, for instance, and therefore the relationship (5b) to be expressed as ${}^{C}D_t^{1/2}\theta(0,t) = -a^{1/2}\partial\theta(0,t)/\partial x$, which is mathematically incorrect [22-24]. However, continuing in this direction with the Dirichlet BC $\theta(0, t)$ = const. this approach provides $a^{1/2}\partial\theta(0,t)/\partial x = 0$. Since $a^{1/2} \neq 0$ the flux at the boundary is $q_0 \equiv \partial\theta(0,t)/\partial x$ and with the Caputo derivative in eq. (6a) it should be zero, *which is unphysical because this result means that the heat does not flow into the medium. Therefore, we got a correct mathematical result which is unphysical only by the fact that RL was mechanistically replaced by the Caputo derivative. With RL, as it is demonstrated further in this article, see eq. (12a) and the text relevant to tab. 1, the flux is* $q_0 \equiv 1/(\pi t)^{1/2}$ *and the heat penetrates the medium.*

The SDIM to HBIM solution (SDIM-1)

With initial condition $\theta(x, t) = 0$, starting from the HBIM relation (2b) and using eq. (5b) to eliminate $\partial\theta(0,t)/\partial x$, with help of eq. (6a) we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta \theta(x,t)\mathrm{d}x = \sqrt{a}\frac{\partial^{1/2}\theta(0,t)}{\partial t^{1/2}} \tag{8}$$

Equation (8) is the principle equation of SDIM-1. Since it was derived by a single integration step and has some restrictions in applications, we will term this approach as SDIM-1 or simple SDIM.

The SDIM to DIM solution (SDIM-2)

The SDIM-2 approach uses a double integration procedure with respect to the space co-ordinate x as in the classical DIM [11, 21, 22], namely:

1st integration

$$\int_0^x \frac{\mathrm{d}}{\mathrm{d}t}\theta(x,t)\mathrm{d}x = -a\frac{\partial\theta(0,t)}{\partial x} \tag{9a}$$

2nd integration

$$\int_0^\delta\int_0^x\left[\frac{\mathrm{d}}{\mathrm{d}t}\theta(x,t)\mathrm{d}x\right]\mathrm{d}x = -a\theta(0,t) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta x\theta(x,t)\mathrm{d}x = -a\theta(0,t) \tag{9b,c}$$

The right-side of eq. (9c) needs the surface temperature to be defined. When the Dirichlet problem is at issue, eq. (9c) is the case of DIM [11]. However, when the flux is specified at $x = 0$ and the surface temperature expression through the approximate profile *should be avoided*, we use a relation coming directly from eqs. (5a) and (5b). Precisely, when the surface temperature is required, as a step of the solution procedure, it can be simply determined by eq. (6a): *this is the second principle step in the* SDIM-2 approach. Finally, from eqs. (9c) and (6c) taking into account that $\partial\theta(0,t)/\partial x = -q_0/\lambda$, we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta x\theta(x,t)\mathrm{d}x = -a\sqrt{a}\,\frac{\partial^{-1/2}}{\partial t^{-1/2}}\left[\frac{\partial\theta(0,t)}{\partial x}\right] \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta x\theta(x,t)\mathrm{d}x = a\sqrt{a}\,\frac{\partial^{-1/2}}{\partial t^{-1/2}}\left(\frac{q_0}{\lambda}\right) \quad \text{(10a,b)}$$

Equation (10b) is the principle relationship of SDIM-2 when a surface flux is specified as a BC.

The method by examples

Prior to developing examples it is worth noting to recall that the time-fractional semi-derivatives and semi-integrals *are used in the right-hand side of the integral relations only* where surface or flux should be expressed *thus avoiding the use of the assumed profile*. The assumed profile, however, has to be used in the BC at $x = 0$.

Example 1: SDIM-1 and constant temperature (concentration) boundary condition at x = 0

The exact solution of this problem is [25]: $\theta_{\mathrm{ex}} = 1 - \mathrm{erf}(\eta/2)$ where $\eta = x/(at)^{1/2}$ is the Boltzmann similarity variable. Assuming a general parabolic profile as $\theta_{\mathrm{a}}^T = \theta_s(1 - x/\delta)^n$, and applying the Goodman BC (3a) and (3b) we get $\theta_{\mathrm{a}}(0,t) = \theta_s = 1$. Replacing θ by the approximate profile θ_{a}^T in the principle equation of SDIM-1, eq. (8) we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta \left(1 - \frac{x}{\delta}\right)^n \mathrm{d}x = \sqrt{a}\,\frac{D^{1/2}}{\partial t^{1/2}}C, \quad C = \mathrm{const.} = 1 \quad (11)$$

Taking into account that RL derivative of a constant C is $\partial^{1/2}C/\partial t^{1/2} = C/(\pi t)^{1/2}$, the integration of eq. (11) with the initial condition $\delta(t = 0) = 0$ yields:

$$\frac{1}{n+1}\frac{\mathrm{d}\delta}{\mathrm{d}t} = \sqrt{a}\,\frac{1}{\sqrt{\pi t}} \Rightarrow \delta_T = \sqrt{at}\,\frac{2(n+1)}{\sqrt{\pi}} \quad \text{(12a,b)}$$

The classical HBIM and DIM, using the same assumed profiles, to the same problem through eq. (2b) and $\partial\theta(0,t)/\partial x = -n/\delta$, provide [10, 11]:

$$\delta_{T(\mathrm{HBIM})} = \sqrt{at}\sqrt{2n(n+1)}, \quad \delta_{T(\mathrm{DIM})} = \sqrt{at}\sqrt{(n+1)(n+2)} \quad \text{(13a,b)}$$

Accuracy of approximation: tests with stipulated exponents n = 2 and n = 3

The plots in fig. 1 reveal that the profiles generated by SDIM-1 are practically indistinguishable from the exact solution. The pointwise errors presented in fig. 2 strongly indicate that the SDIM-1 solutions with either $n = 2$ and $n = 3$ are better than HBIM solutions and in some cases better than DIM profiles (the case of SDIM-1 with $n = 3$). In all the cases, the SDIM-1 solutions with stipulated exponents exhibit pointwise errors less than 0.02.

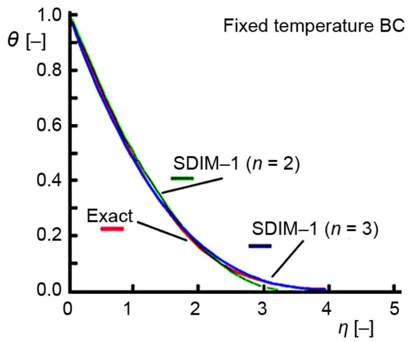

Figure 1. Plots of SDIM-1 solution with stipulated exponents compared to the exact solution

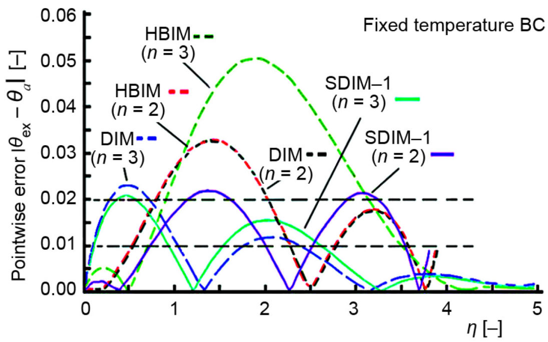

Figure 2. Pointwise errors of various integral-balance solutions with respect to the exact solution; the solid lines correspond to SDIM-1 solutions

Accuracy of approximation: tests with optimized exponents

Since the method developed here and the classic HBIM and DIM are moment methods, the accuracy of approximation depends on the values of the exponent n. The undefined exponent of the profile has been analyzed in [8] and the problem has been further developed towards definition of the optimal exponents of the profile *via* minimization the mean-squared error of approximation [10, 11].

We stress the attention on the fact that the approximate profile satisfies the heat-balance integral but not the original heat conduction equation and therefore the residual function can be defined:

$$\varphi[u_a(x,t)] = \frac{\partial u_a}{\partial t} - a_0 \frac{\partial^2 u_a}{\partial x^2} \tag{14}$$

The residual function $\varphi[u_a(x, t)]$ should be zero if u_a matches the exact solution; otherwise it should attain a minimum for a certain value of the exponent n (the only unspecified parameter of the approximate profile). We will use the definition (14) to find some constraints which the exponent n should obey. With $u_a = (1 - x/\delta)^n$ and $x = 0$, we have $\varphi_T(0, t) = [-n(n - 1)/\delta^2]$. Thus, searching for positive values of n, the heat equation is satisfied for $n > 1$. However, in order to satisfy the Goodman BC $u_a(\delta, t) = \partial u_a(\delta, t)/\partial x = 0$, it is required that $n > 1$. Moreover, for $x \to \delta$ we have $\varphi_T(\delta, t) = \lim_{x\to\delta}\varphi_T(\delta, t) = [-n(n - 1)/\delta^2]\lim_{x\to\delta}(1 - x/\delta)^{n-2}$ and get the condition that the diffusion equation is satisfied at $x = \delta$ when $n > 2$. Therefore, with the previous constraint ($n > 1$) we get the general constraint $n > 2$.

Applying the Langford criterion [2] for the integral-balance method we need:

$$E_{LT}(n,t) = \int_0^\delta \left[\frac{\partial u_a}{\partial t} - a_0 \frac{\partial^2 u_a}{\partial x^2}\right]^2 \mathrm{d}x \to \min \tag{15}$$

Representing the approximation profile $u = (1 - x/\delta)^n$ through the Zener's co-ordinate [26] $\xi = x/\delta$, $0 < \xi < 1$, we get $V(\xi, t) = (1-\xi)^n$ [10, 11]. Hence, the diffusion eq. (1) in ξ-space becomes:

$$-\frac{\mathrm{d}\delta}{\mathrm{d}t}\frac{\xi}{\delta}\frac{\partial V}{\partial \xi} = a_0 \frac{\partial^2 V}{\partial \xi^2}\frac{1}{\delta^2} \tag{16}$$

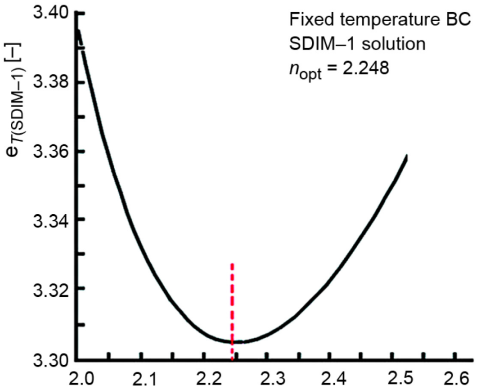

Figure 3. Plots of the function $e_{T(\mathrm{SDIM-1})}(n)$ defining the optimal exponent $n \approx 2.248$ SDIM-1 solution of the fixed temperature BC problem

Further, the expression about δ_T is (12b) and the product $\delta(\mathrm{d}\delta/\mathrm{d}t) = 2a_0(n+1)^2/\pi$ is time independent. Then, after the integration from 0 to $\xi = x/\delta = 1$ the error measure $E_{LT}(n, t)$ can be expressed as a ratio $E_{LT(\mathrm{SDIM-1})} = e_{T(\mathrm{SDIM-1})}(n)/\delta^4$ [10, 11]. Therefore, the squared error of minimization $E_{LT}(n, t)$ decays rapidly in time and searching or the optimal exponent of the profile the procedure focuses on the minimization of $e_{T(\mathrm{SDIM-1})}(n)$ the nominator with respect to n. The procedure is well described for the linear eq. (11) [10, 11] and for non-linear problems in [27]. The function $e_{T(\mathrm{SDIM-1})}(n)$, see eq. (17) and fig. 3.

$$e_{T(\mathrm{SDIM-1})}(n) = \frac{0.405437(n+1)^2}{n(2n-1)(2n+1)} + \frac{n(n-1)}{2n-3} + \frac{1.273479n(n-1)(n+1)}{(2n-1)(2n+1)} \tag{17}$$

The rational function $e_{T(\mathrm{SDIM-1})}(n)$ exhibits the only physically realistic minimum at $n \approx 2.248$ (fig. 3) resulting in $E_{LT(\mathrm{SDIM-1})} \approx 0.01834$.

The numerical simulations of the approximate SDIM-1 solution and the exact one are shown in fig. 4(a). The pointwise errors of HBIM, DIM and SDIM-1 solutions, fig. 4(b), with optimal exponents reveal that SDIM-1 exhibits better accuracy compared to the behaviour of HBIM and DIM approximations. The effect is due to the fact that the key step in the SDIM-1 solution: *avoiding the approximation of space derivative at x = 0 through the assumed profile*.

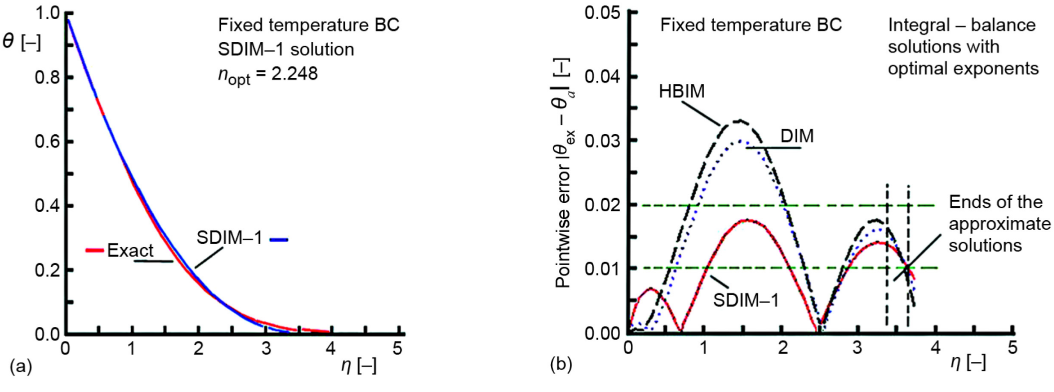

Figure 4. (a) SDIM-1 solution with optimal exponent compared to the exact solution; fixed temperature BC problem, (b) pointwise errors of various integral-balance solutions (with optimal exponents) with respect to the exact solution; the solid line corresponds to the SDIM-1 solutions

Moreover, as an expected results the penetration depths corresponding to $\theta_a = 0$, that is η_{final}, with the optimal exponents are: $\delta_{T(\mathrm{SDIM-1})} \approx 3.665$, $\delta_{T(\mathrm{HBIM})} \approx 3.475$, and $\delta_{T(\mathrm{DIM})} \approx 3.471$. Hence, with SDIM-1 *we get the longest penetration depth that affects the approximation error close to the end of the solution*.

Surface flux approximation

Moreover, the dimensionless surface flux, Q, can be expressed as $Q = q_0(at)^{1/2}/\lambda[\theta(0, t)]$. The estimations of Q obtained by different integration techniques are summarized in tab. 1. The exact solution [25] is $Q_{exact} = 1/\pi^{1/2}$. The data in tab. 1 indicate that in all the cases the approximate solutions overestimate the surface heat flux.

Table 1. Evaluation of the dimensionless surface flux $Q = q_0(at)^{1/2}/\lambda[\theta(0, t)]$ for various integration approaches and fixed temperature BC

	Solutions with stipulated exponents				Solutions with optimal exponents		
Q	HBIM	DIM	SDIM-1	Exact	HBIM	DIM	SDIM-1
$n = 2$	0.577 (+0.023)	0.577 (+0.023)	0.590 (+0.046)	0.564	0.577 (+0.023)	0.586 (+0.039)	0.613 (+0.087)
$n = 3$	0.612 (+0.085)	0.670 (+0.085)	0.663 (+0.117)		$n_{opt} \approx 2.008$	$n_{opt} \approx 2.074$	$n_{opt} \approx 2.248$

Note: the data in the brackets are the approximation errors

Moreover, despite the lower pointwise errors of SDIM-1 with either stipulated or optimized exponents, better approximations of the surface flux at $x = 0$ are provided by HBIM and DIM. Hence, if we need low approximation error across the penetration depth and close to its front, then SDIM-1 is the adequate solution. Otherwise, if the surface flux is the target, then HBIM or DIM should be used.

Example 2: SDIM-2 and fixed flux boundary condition at x = 0

With the BC $-\lambda[\partial\theta(0, t)/\partial x] = q_0$ the exact solution of eq. (1) is [25]:

$$\theta_{ex}(x,t) = 2\frac{q_0}{\lambda}\sqrt{at}\left[\frac{1}{\sqrt{\pi}}\exp\left(-\frac{x^2}{4at}\right) - \frac{x}{\sqrt{at}}\mathrm{erf}c\left(\frac{x}{2\sqrt{at}}\right)\right]$$

$$\theta_{ex}(0,t) = 2\frac{q_0}{\lambda}\frac{\sqrt{at}}{\sqrt{\pi}} \approx 1.1283\frac{q_0}{\lambda}\sqrt{at} \tag{18a,b}$$

The classical HBIM, eq. (2b), needs in its right hand side the gradient at $x = 0$ to be defined. In this case no additional refinements are required since the BC is $[\partial\theta(0, t)/\partial x] = q_0/\lambda$. The result is $\delta_{q(HBIM)} = (at)^{1/2}[n(n + 1)]^{1/2}$ [10, 16]. For that reason, it is more challenging to see how the solution of the DIM approach could be improved by applying the SDIM-2 approach.

With assumed profile $\theta_a^q = \theta_s^q(1 - x/\delta)^n$ and applying the BC (3a) and (3b) we get:

$$\theta_a^q = \frac{q_0}{\lambda}\frac{\delta}{n}\left(1 - \frac{x}{\delta}\right)^n \tag{19}$$

In the basic DIM°, eq. (9c), where the product $a\theta(0, t)$ balances the double heat-balance integral the surface temperature $\theta(0, t)$ *is approximated through the assumed profile* setting $x = 0$, *i. e.* we use $a\theta_a(0, t) = a(q_0/\lambda)(\delta/n)$. Then, the result is the classical DIM solution [10, 11]:

$$\frac{1}{n(n+1)(n+2)}\frac{\mathrm{d}\delta^3}{\mathrm{d}t} = a\frac{\delta}{n} \Rightarrow \delta_{q(\mathrm{DIM})} = \sqrt{at}\sqrt{(n+1)(n+2)\frac{2}{3}} \tag{20a,b}$$

Alternatively, *if the use of the assumed profile in the definition of the surface temperature in the right-hand side of the integral relation is avoided*, then following the SDIM-2 eq. (10b) we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta \frac{q_0}{\lambda}x\frac{\delta}{n}\left(1-\frac{1}{\delta}\right)^n \mathrm{d}x = a\sqrt{a}\frac{D^{-1/2}}{\partial t^{-1/2}}\left(\frac{q_0}{\lambda}\right), \quad D_t^{-1/2}\left(\frac{q_0}{\lambda}\right) = \left(\frac{q_0}{\lambda}\right)\left(\frac{2\sqrt{t}}{\sqrt{\pi}}\right) \tag{21a,b}$$

Therefore, from eq. (21a) we get:

$$\frac{1}{n(n+1)(n+2)}\frac{\mathrm{d}\delta^3}{\mathrm{d}t} = a^{\frac{3}{2}}\frac{2\sqrt{t}}{\sqrt{\pi}} \Rightarrow \delta_q^{s2} = \sqrt{at}\left[\frac{4}{3}\frac{n(n+1)(n+2)}{\sqrt{\pi}}\right]^{\frac{1}{3}}, \ \delta_q^{s2} = \sqrt{at}F_{s2}(n) \tag{22a,b,c}$$

The difference between $\delta_{\mathrm{q(DIM)}}$, eq. (20b), and δ_q^{s2}, eq. (22b) is in the function $F(n)$ depending on the exponent n and the integration approach applied. Moreover, the dimensionless approximate profiles expressed in terms of the similarity variable $\eta = x/(at)^{1/2}$ as $\Theta_a^q(\eta) = \theta_a^q(\eta)\lambda/q_0(at)^{1/2}$ allow comparing them to the exact solution. Therefore, we have $\Theta_a(\eta) = [F_s(n)/n][1 - \eta/F_s(n)]^n$.

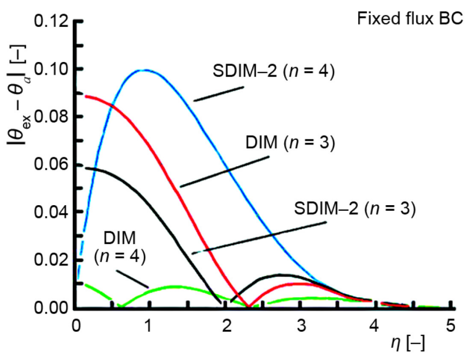

Figure 5. Pointwise errors of integral-balance solutions developed by different integration techniques and stipulated integer exponents

Accuracy of approximation: tests with stipulated integer exponents

Tests demonstrating pointwise approximation errors with the SDIM-2 solution when the exponent n is stipulated, as in the classical integral-balance solutions [1, 16], are presented in fig. 5. The numerical experiments were performed with $n = 3$ and $n = 4$ because the optimal exponents (discussed in the next section) are between these two integer values. We do no show the plots of the approximate solutions since they are undistinguishable visually and therefore, we stress the attention on the approximation errors. The pointwise errors (fig. 5) indicate that, in general, better accuracy is provided by the DIM solution with $n = 4$ over the entire range of variations of η. Especially for the SDIM-2 solution with either $n = 3$ or $n = 4$, the acceptable approximation errors are in the range $2.5 < \eta < 4.5$, but in both cases the surface temperature approximation is unacceptable.

Accuracy of approximation: tests with optimized exponents

Similar to Example 1, the residual error function can be constructed for the case of fixed flux BC. We skip the details in the expression of the residual function since it is available elsewhere [11, 27, 28]. Minimization of the squared error of approximation represented as

$E_{LQ}(n,\ t) = e_{LQ}(n)/\delta^4$ over the penetration depth provided $n_{\text{SDIM–2}}(\text{opt}) \approx 3.979$ and $E_{LQ}(n) \approx 0.0257$. At the same time, the optimal exponents of HBIM and DIM to the same problems are $n^q_{\text{HBIM}}(\text{opt}) \approx 3.535$ and $n^q_{\text{DIM}}(\text{opt}) \approx 3.798$. These are expected differences because the dimensionless functions $F_n(n)$ multiplying the length scale $(at)^{1/2}$ depend on the integration method applied.

The plots in fig. 6(a) reveal that the performance of SDIM-2 is better than that DIM. Since the HBIM is the correct approach rather than DIM, as commented, the fact that the SDIM-2 solution is comparable to that of HBIM confirms additionally the appropriateness of the approach used here. Since the determination of the optimal exponent is based on the minimization of $e_{LQ}(n)$ it is quite informative to demonstrate what the solution behaviour is when the exponent of the SDIM-2 solution varies around $n_{\text{SDIM–2}}(\text{opt}) \approx 3.979$, as it is shown in fig. 6(b). For $n < 3.979$ the solutions have unacceptable approximation errors when $\eta < 2$ but for larger η the accuracy is better that the case when the optimal exponent is used. Similarly, when $n > 3.979$ worst approximations are exhibited for $\eta < 1$ while the accuracies better that the optimal solution can be attained for larger η.

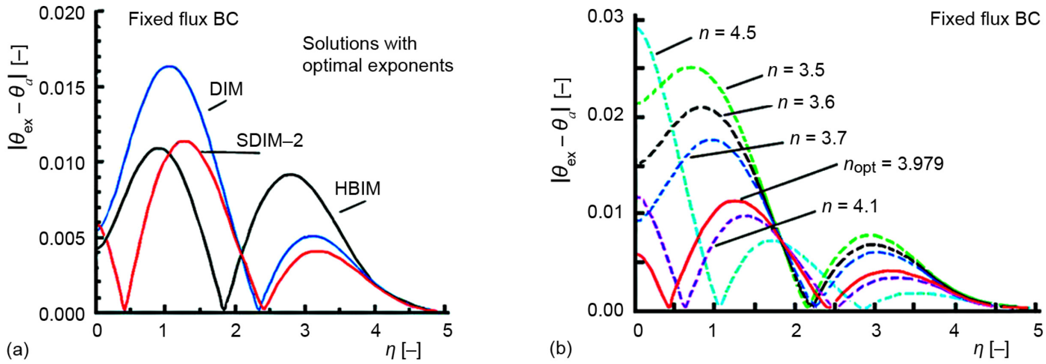

Figure 6. Pointwise errors of integral balance solutions with optimal exponents of the parabolic profile; (a) comparison of HBIM, DIM, and SFIM-2, (b) comparison of pointwise errors of SDIM-2 solutions when the exponent varies around the optimal value

In addition, as an expected results the penetration depths corresponding to $\theta^q_{\text{a}} = 0$, that is η_{final}, with the optimal exponents are: $\delta_{q(\text{SDIM–2})} \approx 4.4666$, $\delta_{q(\text{HBIM})} \approx 4.0039$, and $\delta_{q(\text{DIM})} \approx 4.3064$. Hence, with SDIM-2 *we get the longest penetration depth that affects the approximation error close to the end of the solution*.

Surface temperature approximation.

As a next step of evaluation of the method, the dimensionless surface temperature, Θ_s, can be expressed as $\Theta_\text{s} = \lambda\theta(0,\ t)/q_0(at)^{1/2}$. Thus we may estimate Θ obtained by different integration techniques (tab. 2). The exact solution [25] is $Q_{\text{s–exact}} = 2/\pi^{1/2}$.

The data in tab. 2 (the columns to the left of the exact solution) reveal that better approximation is provided by the HBIM solution, which confirms the previous comments. Further, using the optimal exponents, the SDIM-2 solution overestimates the surface temperature, while HBIM and DIM underestimate it. However, the approximation errors are of one and the same order of magnitude.

Table 2. Evaluation of the dimensionless surface temperature $\Theta_s = \lambda\theta(0, t)/q_0(at)^{1/2}$ for various integration approaches and constant flux BC

Θ	HBIM	DIM	SDIM-2	Exact	HBIM	DIM	SDIM-2
	Solutions with stipulated exponents				Solutions with optimal exponents		
$n = 3$	1.154 (+0.089)	1.217 (+ 0.089)	1.186 (–0.094)	1.128	1.132 (–0.004)	1.133 (–0.005)	1.122 (+0.006)
$n = 4$	1.118 (–0.010)	1.118 (–0.01)	1.121 (+0.05)		$n_{\text{opt}} \approx 3.563$	$n_{\text{opt}} \approx 3.798$	$n_{\text{opt}} \approx 3.979$

Note: the data in the brackets are the approximation errors

Brief analysis of the method developed and conclusions

At the end of this article we have to emphasize on some principle points of the developed solutions, among them:

- The examples demonstrated a new approach in solutions of transient heat conduction problems by the Goodman integral balance method where the principle step is the use of fractional-time semi-derivative of RL sense to relate the heat flux and the surface temperature at the front ($x = 0$) of the heated medium.
- Looking precisely at the integration techniques used so far with the Goodman method (commented in the Introduction) we may define two principle groups:
  (1) *Single equation approach.* This is, in fact, the classical Goodman's' HBIM expressed by eq. (2b). The main problem emerging in application of eq. (4) is that the gradient at $x = 0$ should be expressed through the preliminary defined approximate profile. This principle drawback of HBIM can be avoided by the DIM resulting in eq. (9c). However, with DIM we face a similar problem when the surface flux is prescribed at $x = 0$. To this point, we have to emphasize on the fact that eqs. (2) and (4) express *the zeroth moment* of eq. (1), while the left-side of the DIM equation, eq. (9c) is *the first moment of θ.*
  (2) *Two equations approach.* The philosophy of these solutions bring into play *the zeroth moment* expressed by eq. (2b), see also eq. (23a), and a *moment-like* equation (developed by multiplication of eq. (1) by θ and integration over the penetration depth, see eq. (23b), namely:

$$\int_0^\delta \frac{\partial\theta}{\partial t}\,\mathrm{d}x = a\int_0^\delta \frac{\partial^2\theta}{\partial x^2}\,\mathrm{d}x \quad \text{and} \quad \int_0^\delta \theta\frac{\partial\theta}{\partial t}\,\mathrm{d}x = a\int_0^\delta \theta\frac{\partial^2\theta}{\partial x^2}\,\mathrm{d}x \tag{23a,b}$$

Introducing the approximate profile in both eqs. (23a) and (23b) and solving them simultaneously we get *a set of two equations with two unknowns*: the penetration depth δ and the gradient $\partial\theta(0, t)/\partial x$. Classical examples of this approach are the problems solved by Zien [29, 30] despite the fact that approximate profile used in these works differs from that used in the present work.

Consequently, the reasonable question is: To which group of integration methods the developed SDIM approach belongs? Certainly, the conceived SDIM is a *two-equation method*. *The first equation* is either the classical HBIM equation (zeroth moment) of DIM equation (first moment). *The second equation is eq. (6a) relating the flux and the temperature at any point of the medium*; *it is especially used at boundary* $x = 0$, see eq. (6b). This point is a quite important element of the SDIM solution because the use of the approximate profile in presentation of the right-hand side of the integral-balance equations is completely avoided.

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