

ON TRIPLY DIFFUSIVE CONVECTION IN COMPLETELY CONFINED FLUIDS

by

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Original scientific paper
<https://doi.org/10.2298/TSCI150304214P>

The present paper carries forward Prakash et al. [21] analysis for triple diffusive convection problem in completely confined fluids and derives upper bounds for the complex growth rate of an arbitrary oscillatory disturbance which may be neutral or unstable through the use of some non-trivial integral estimates obtained from the coupled system of governing equations of the problem.

Key words: triple diffusive convection, oscillatory motion, Lewis number, concentration Rayleigh number, completely confined fluids

Introduction

Convective motions can occur in a stably stratified fluid when there are two components contributing to the density which diffuse at different rates. This phenomenon is called double-diffusive convection. To determine the conditions under which these convective motions will occur, the linear stability of two superposed concentration (or one of them may be temperature gradient) gradients has been studied by Stern [1], Veronis [2], Nield [3], Baines and Gill [4], Turner [5], Akrouf *et al.* [6], El-Maghlany *et al.* [7], Periyanaagounder *et al.* [8], Aggarwal and Makhija [9] and Aggarwal and Verma [10].

The case of two component system has been considered only. However, it has been recognized later (Griffiths [11], Turner [12]) that there are many situations wherein more than two components are present. Examples of such multiple diffusive convection fluid systems include the solidification of molten alloys, geothermally heated lakes, magmas and their laboratory models, and sea water. For the detailed overview of the work done on triply/multiple diffusive convection one may refer to Griffiths [11], Pearlstein *et al.* [13], Lopez *et al.* [14], Ryzhkov and Shevtsova [15, 16], Rionero [17, 18], Prakash *et al.* [19, 20]. These researchers found that small concentrations of a third component with a smaller diffusivity can have a significant effect upon the nature of diffusive instabilities and direct salt finger and oscillatory modes are simultaneously unstable under a wide range of conditions, when the density gradients due to components with the greatest and smallest diffusivity are of same signs. Recently Prakash *et al.* [21] derived the bounds for the complex growth rate in triply diffusive convection.

All these researchers have confined themselves to horizontal layer geometry, perhaps, due to the complexity involved in the analysis of the hydrodynamic problems with arbitrary geometries. However, there are a few researchers (Sherman and Ostrach [22], Gupta *et al.*

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[23, 24], Gupta and Dhiman [25], Mohan *et al.* [26]) who have extended the classical work to more general hydrodynamic stability problems with arbitrary boundaries. In the present communication, which is motivated by the desire to extend the works of Gupta *et al.* [23] to more complex problem, namely, triply diffusive convection problem for completely confined fluids and bounds for the complex growth rate are obtained which are important keeping in view the fact that exact solutions, even in the case of simple horizontal plane rigid boundaries, are not obtainable in a closed form. The following result is obtained in this direction.

The complex growth rate $p = p_r + ip_i$ of an arbitrary oscillatory ($p_i \neq 0$) disturbance, which may be neutral or unstable for triply diffusive convection problem in completely confined fluid lies inside a semicircle with centre origin and radius equals $[(R_1 + R_2)\sigma]^{1/2}$ in the right half of the complex p -plane. The results of Gupta *et al.* [23] for double-diffusive convection problem in completely confined fluids are obtained as a consequence.

Mathematical formulation and analysis

Consider a Boussinesq fluid statically confined in an arbitrary completely enclosed region, fig. 1, which is maintained at a uniform temperature and concentration gradient parallel to the body force acting on a fluid by applying certain prescribed thermal and concentration boundary conditions on the bounding walls. The problem under investigation is to examine the stability of this physical configuration when the heat and

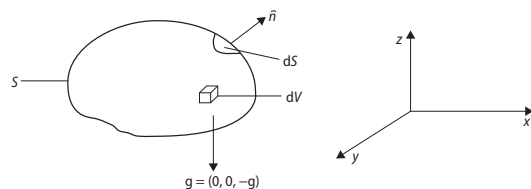


Figure 1. Physical configuration

the two concentrations make opposing contributions to the vertical density gradient. It is further assumed that the cross diffusion effects can be neglected.

The governing linearized perturbation equations in non-dimensional form for the problem with time dependence of the form $\exp(pt)$ ($p = p_r + ip_i$ being complex in general) are given by Gupta *et al.* [23]

$$\frac{p}{\sigma} \bar{U} = \nabla^2 \bar{U} - \text{grad}(P) + R\theta \hat{k} - R_1 \phi_1 \hat{k} - R_2 \phi_2 \hat{k} \quad (1)$$

$$\nabla^2 \theta - p\theta = -\bar{U} \cdot \hat{k} \quad (2)$$

$$\tau_1 \nabla^2 \phi_1 - p\phi_1 = -\bar{U} \cdot \hat{k} \quad (3)$$

$$\tau_2 \nabla^2 \phi_2 - p\phi_2 = -\bar{U} \cdot \hat{k} \quad (4)$$

$$\text{div} \bar{U} = 0 \quad (5)$$

The equations have been written in dimensionless forms by using the scale factors κ/d , d^2/κ , βd , $(\rho v \kappa)/d^2$, $\beta_1 d$, and $\beta_2 d$ for velocity, time, temperature, pressure, and the two concentrations, respectively. The symbols which appear in this paragraph and in eqs. (1) and (2) have been defined in the nomenclature.

We seek solutions of eqs. (1)-(5) in a simply connected subset V of R^3 with boundary S subject to the following homogeneous time independent boundary conditions:

$$\bar{U} = 0 = \theta = \phi_1 = \phi_2 \quad (6)$$

on S (rigid bounding surface with fixed temperature and mass concentrations).

Equations (1)-(5) together with boundary conditions (6) constitute an eigenvalue problem for p for prescribed values of the other parameters and the system is stable, neutral or unstable according as p_r is negative, zero or positive. Further if $p_r = 0$ implies $p_i = 0$, then the principle of the exchange of stabilities is valid otherwise we will have overstability.

Now we prove the following theorem.

Theorem: If $R > 0$, $R_1 > 0$, $R_2 > 0$, $p_r \geq 0$, and $p_i \neq 0$, then a necessary condition for the existence of a non-trivial solution $(p, \bar{U}, \theta, \phi_1, \phi_2)$ of eqs. (1)-(5) together with boundary conditions (6) is that:

$$|p|^2 < (R_1 + R_2)\sigma$$

Proof: We rewrite system of eqs. (1)-(4) in the following convenient forms:

$$\frac{p}{\sigma} \bar{U} + \text{grad}(P) - \nabla^2 \bar{U} - R\theta \hat{k} + R_1 \phi_1 \hat{k} + R_2 \phi_2 \hat{k} = 0 \quad (7)$$

$$-R[\nabla^2 \theta - p\theta + \bar{U} \cdot \hat{k}] = 0 \quad (8)$$

$$R_1[\tau_1 \nabla^2 \phi_1 - p\phi_1 + \bar{U} \cdot \hat{k}] = 0 \quad (9)$$

$$R_2[\tau_2 \nabla^2 \phi_2 - p\phi_2 + \bar{U} \cdot \hat{k}] = 0 \quad (10)$$

Forming the dot product of eq. (7) with \bar{U}^* (* indicates complex conjugation) and integrating over the domain V , we get:

$$\begin{aligned} & \frac{p}{\sigma} \int_V (\bar{U} \cdot \bar{U}^*) dV + \int_V (\text{grad}P) \cdot \bar{U}^* dV - \int_V (\bar{U}^* \cdot \nabla^2 \bar{U}) dV - \\ & - R \int_V [(\theta \hat{k}) \cdot \bar{U}^*] dV + R_1 \int_V [(\phi_1 \hat{k}) \cdot \bar{U}^*] dV + R_2 \int_V [(\phi_2 \hat{k}) \cdot \bar{U}^*] dV = 0 \end{aligned} \quad (11)$$

Subsequently, for convenience in writing, we omit V and the infinitesimal volume dV from the integral sign and the integrand, respectively.

Multiplying eqs. (8)-(10) by θ^* , ϕ_1^* , ϕ_2^* , respectively, integrating over the domain V , we get:

$$-R \int \theta^* [\nabla^2 \theta - p\theta + \bar{U} \cdot \hat{k}] = 0 \quad (12)$$

$$R_1 \int \phi_1^* [\tau_1 \nabla^2 \phi_1 - p\phi_1 + \bar{U} \cdot \hat{k}] = 0 \quad (13)$$

$$R_2 \int \phi_2^* [\tau_2 \nabla^2 \phi_2 - p\phi_2 + \bar{U} \cdot \hat{k}] = 0 \quad (14)$$

Now, adding eqs. (12)-(14) to eq. (11), we have:

$$\begin{aligned} & \frac{p}{\sigma} \int (\bar{U} \cdot \bar{U}^*) + \int (\text{grad}P) \cdot \bar{U}^* - \int (\bar{U}^* \cdot \nabla^2 \bar{U}) - R \int \theta^* (\nabla^2 - p)\theta + R_1 \int \phi_1^* (\tau_1 \nabla^2 - p)\phi_1 + \\ & + R_2 \int \phi_2^* (\tau_2 \nabla^2 - p)\phi_2 - RI + R_1 I_1 + R_2 I_2 = 0 \end{aligned} \quad (15)$$

where

$$I = 2Re\left[\int(\theta\hat{k} \cdot \bar{U}^*)\right], \quad I_1 = 2Re\left[\int(\phi_1\hat{k} \cdot \bar{U}^*)\right], \quad I_2 = 2Re\left[\int(\phi_2\hat{k} \cdot \bar{U}^*)\right]$$

and Re denotes the real part. Using Gauss theorem and boundary conditions (6), we have:

$$\int(\text{grad } P) \cdot \bar{U}^* = \int_S P \bar{U}^* \cdot \hat{n} - \int P \text{div} \bar{U}^* = 0 \quad (16)$$

$$\begin{aligned} \int(\bar{U}^* \cdot \nabla^2 \bar{U}) &= -\int(\text{curl curl} \bar{U} \cdot \bar{U}^*) \\ &= -\int \text{curl} \bar{U} \cdot \text{curl} \bar{U}^* - \int_S (\text{curl} \bar{U}) \times \bar{U}^* \cdot \hat{n} \\ &= -\int \text{curl} \bar{U} \cdot \text{curl} \bar{U}^* \end{aligned} \quad (17)$$

$$\begin{aligned} \int(\theta^* \nabla^2 \theta) &= \int_S (\theta^* \nabla \theta) \cdot \hat{n} - \int \nabla \theta \cdot \nabla \theta^* \\ &= -\int \nabla \theta \cdot \nabla \theta^* \end{aligned} \quad (18)$$

$$\begin{aligned} \int(\phi_1^* \nabla^2 \phi_1) &= \int_S (\phi_1^* \nabla \phi_1) \cdot \hat{n} - \int \nabla \phi_1 \cdot \nabla \phi_1^* \\ &= -\int \nabla \phi_1 \cdot \nabla \phi_1^* \end{aligned} \quad (19)$$

$$\begin{aligned} \int(\phi_2^* \nabla^2 \phi_2) &= \int_S (\phi_2^* \nabla \phi_2) \cdot \hat{n} - \int \nabla \phi_2 \cdot \nabla \phi_2^* \\ &= -\int \nabla \phi_2 \cdot \nabla \phi_2^* \end{aligned} \quad (20)$$

where \hat{n} is a unit outward drawn normal at any point on S . Using integral relations (16)-(20) in eq. (15), we have:

$$\begin{aligned} \frac{p}{\sigma} \int(\bar{U} \cdot \bar{U}^*) + \int \text{curl} \bar{U} \cdot \text{curl} \bar{U}^* + R \int(\nabla \theta \cdot \nabla \theta^* + p|\theta|^2) - R_1 \int(\tau_1 \nabla \phi_1 \cdot \nabla \phi_1^* + p|\phi_1|^2) - \\ - R_2 \int(\tau_2 \nabla \phi_2 \cdot \nabla \phi_2^* + p|\phi_2|^2) - RI + R_1 I_1 + R_2 I_2 = 0 \end{aligned} \quad (21)$$

Equating the imaginary part of eq. (21) to zero, we have for $p_i \neq 0$:

$$\frac{1}{\sigma} \int(\bar{U} \cdot \bar{U}^*) + R \int|\theta|^2 = R_1 \int|\phi_1|^2 + R_2 \int|\phi_2|^2 \quad (22)$$

Multiplying eq. (3) by its complex conjugate, integrating over V and using integral relation (19), we have:

$$\tau_1^2 \int|\nabla^2 \phi_1|^2 + |p|^2 \int|\phi_1|^2 + 2p_r \tau_1 \int \nabla \phi_1 \cdot \nabla \phi_1^* = \int|\bar{U} \cdot \hat{k}|^2 \quad (23)$$

Since $p_r \geq 0$, $p_i \neq 0$, eq. (23) implies that:

$$\int|\phi_1|^2 < \frac{1}{|p|^2} \int|\bar{U} \cdot \hat{k}|^2 \leq \frac{1}{|p|^2} \int \bar{U} \cdot \bar{U}^* \quad (24)$$

In the same manner, it follows from eq. (4) that:

$$\int |\phi_2|^2 \leq \frac{1}{|p|^2} \int \bar{\mathbf{U}} \cdot \mathbf{U}^* \quad (25)$$

Utilizing inequalities (24) and (25) in eq. (22), we have:

$$\left(\frac{1}{\sigma} - \frac{R_1}{|p|^2} - \frac{R_2}{|p|^2} \right) \int \bar{\mathbf{U}} \cdot \mathbf{U}^* + R \int |\theta|^2 < 0 \quad (26)$$

which clearly implies that:

$$|p|^2 < (R_1 + R_2)\sigma$$

This proves the *theorem*.

The previous theorem can be stated from the physical point of view as the complex growth rate $p = p_r + ip_i$ of an arbitrary oscillatory ($p_i \neq 0$) disturbance, which may be neutral or unstable for triply diffusive convection problem in completely confined fluid lies inside a semicircle with centre origin and radius equals $[(R_1 + R_2)\sigma]^{1/2}$ in the right half of the complex p - plane.

Special cases: The following results may be obtained from *theorem* as special cases:

- for thermohaline convection of Veronis [2] type in completely confined fluids ($R > 0, R_1 > 0, R_2 = 0$), $|p| < (R_1\sigma)^{1/2}$, Gupta *et al.* [23],
- for thermohaline convection of Stern [1] type in completely confined fluids ($R < 0, R_1 < 0, R_2 = 0$), $|p| < (|R|\sigma)^{1/2}$, Gupta *et al.* [23], and
- for triply diffusive convection analogous to Stern [1] type in completely confined fluids ($R < 0, R_1 < 0, R_2 < 0$), $|p| < (|R|\sigma)^{1/2}$.

Proof: Putting $R_1 = -|R_1|$ and $R_2 = |R_2|$ in eq. (1), and adopting the same procedure as is used to prove *theorem*, we obtain the desire result.

Conclusion

Upper bounds for the complex growth rate of an arbitrary oscillatory disturbance which may be neutral or unstable for triple diffusive convection problem in completely confined fluids through the use of some non-trivial integral estimates are obtained from the coupled system of governing equations of the problem. These bounds are important since exact solutions in the closed form are not obtainable for the present problem. Further the existing results of thermohaline convection problem in completely confined fluids are obtained as a consequence.

Nomenclature

d - depth of layer, [m]
 g - acceleration due to gravity, [ms^{-2}]
 P - pressure, [Pa]
 p - growth rate, [s^{-1}]
 R - thermal Rayleigh number, [-]
 R_1 - solutal Rayleigh number for first concentration component, [-]
 R_2 - solutal Rayleigh number for second concentration component, [-]
 t - time, [s]

$\bar{\mathbf{U}}$ - velocity, [ms^{-1}]

Greek symbols

β - uniform temperature gradient [Km^{-1}]
 β_1 - uniform concentration gradient for first concentration component, [-]
 β_2 - uniform concentration gradient for second concentration component, [-]
 θ - perturbation in temperature, [K]
 κ - thermal diffusivity, [m^2s^{-1}]

κ_1 – mass diffusivities of first concentration component, [m^2s^{-1}]	τ_1 – Lewis number for first concentration component, ($= \kappa_1/\kappa$), [–]
κ_2 – mass diffusivities of second concentration component, [m^2s^{-1}]	τ_2 – Lewis number for second concentration component, ($= \kappa_2/\kappa$), [–]
ν – kinematic viscosity, [m^2s^{-1}]	ϕ_1 – perturbation concentration of first component, [kg]
σ – Prandtl number, ($= \nu/\kappa$), [–]	ϕ_2 – perturbation concentration of second component, [kg]
ρ – density, [kgm^{-3}].	

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