A NOVEL SCHEDULE FOR SOLVING THE TWO-DIMENSIONAL DIFFUSION PROBLEM IN FRACTAL HEAT TRANSFER

by

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In this work, the local fractional variational iteration method is employed to obtain approximate analytical solution of the two-dimensional diffusion equation in fractal heat transfer with help of local fractional derivative and integral operators.

Key words: *variational iteration method, diffusion equation, fractal heat transfer, local fractional derivative*

Introduction

In this present paper we consider the 2-D diffusion problem in fractal heat transfer involving local fractional derivatives (LFD) [1, 2]:

$$\nabla^{2\alpha}u(x,y,t)-\frac{1}{D^{\alpha}}\frac{\partial^{\alpha}u(x,y,t)}{\partial t^{\alpha}}=0 \tag{1}$$

where the local fractional Laplace operator [1, 3, 4] reads as:

$$\nabla^{2\alpha}=\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}}{\partial y^{2\alpha}} \tag{2}$$

and D^{α} denotes the fractal diffusion constant which is, in essence, a measure for the efficiency of the spreading of the underlying substance. Recently, 1-D diffusion problems were studied by several authors by using local fractional decomposition method [5], series expansion [6], variational iteration [7], and functional [8] methods. The local fractional variational iteration method (LFVIM) structured in [9] was adopted to deal with silk cocoon [10], damped and dissipative wave [11], Laplace [12], wave on Cantor sets [13], and parabolic Fokker--Planck [14], and other [15, 16]. Aim of the paper is to utilize the technology to solve the problem (1).

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LFVIM for 2-D diffusion in fractal heat transfer

Let the LFD of $f(x)$ of order α at the point $x = x_0$ be given [1-16] as:

$$D_x^\alpha f(x_0) = \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} f(x)\Big|_{x=x_0} = f^{(\alpha)}(x_0) = \lim_{x\to x_0} \frac{\Delta^\alpha [f(x) - f(x_0)]}{(x - x_0)^\alpha} \tag{3}$$

where $\Delta^\alpha [f(x) - f(x_0)] \cong \Gamma(\alpha + 1)[f(x) - f(x_0)]$.

Let local fractional integral of $f(x)$ in the interval $[a, b]$ be defined [1-16] as:

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(\mathrm{d}t)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t\to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha \tag{4}$$

with the partition of the interval $[a, b]$, $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_0, \Delta t_1, ...\}$, and $j = 0,\ldots, N-1$, $t_0 = a$, and $t_N = b$.

According to the theory of LFVIM, [9-16], we write the iteration formula as:

$$u_{n+1}(x,y,t) = u_n(x,y,t) +$$

$$+ {}_0I_t^{(\alpha)} \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \{L_\xi^{(\alpha)} u_n(x,y,\xi) - D^\alpha [L_{xx}^{(2\alpha)} \tilde{u}_n(x,y,\xi) + L_{yy}^{(2\alpha)} \tilde{u}_n(x,y,\xi)]\} \tag{5}$$

where $\lambda^\alpha / \Gamma(1+\alpha)$ is a Lagrange multiplier, $L_\xi^{(\alpha)} = \dfrac{\partial^\alpha}{\partial t^\alpha}$, $L_{xx}^{(2\alpha)} = \dfrac{\partial^{2\alpha}}{\partial x^{2\alpha}}$, and $L_{yy}^{(2\alpha)} = \dfrac{\partial^{2\alpha}}{\partial y^{2\alpha}}$.

Taking local fractional variation of eq. (5), we present:

$$\delta^\alpha u_{n+1}(x,y,t) = \delta^\alpha u_n(x,y,t) +$$

$$+ \delta^\alpha {}_0I_t^{(\alpha)} \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \{L_\xi^{(\alpha)} u_n(x,y,\xi) - D^\alpha [L_{xx}^{(2\alpha)} \tilde{u}_n(x,y,\xi) + L_{yy}^{(2\alpha)} \tilde{u}_n(x,y,\xi)]\} = 0 \tag{6}$$

This yields the stationary conditions:

$$1 + \frac{\lambda^\alpha}{\Gamma(1+\alpha)}\Bigg|_{\xi=t} = 0, \quad \left(\frac{\lambda^\alpha}{\Gamma(1+\alpha)}\right)^{(\alpha)}\Bigg|_{\xi=t} = 0 \tag{7a, b}$$

This in turn gives Lagrange multiplier:

$$\frac{\lambda^\alpha}{\Gamma(1+\alpha)} = -1 \tag{7c}$$

Substituting this value of the Lagrange multiplier into eq. (5), one gives the iteration formula:

$$u_{n+1}(x,y,t) = u_n(x,y,t) -$$

$$- {}_0I_t^{(\alpha)} L_\xi^{(\alpha)} u_n(x,y,\xi) - D^\alpha \{L_{xx}^{(2\alpha)} u_n(x,y,\xi) + L_{yy}^{(2\alpha)} u_n(x,y,\xi)\} \tag{8}$$

We now consider the initial condition of eq. (1), namely:

$$u(x,y,0) = E_\alpha(x^\alpha)E_\alpha(y^\alpha) \tag{9}$$

Start with the zeroth approximation:

$$u_0(x,y,t) = u(x,y,0) = E_\alpha(x^\alpha)E_\alpha(y^\alpha) \tag{10}$$

Substituting eq. (10) into eq. (8) we obtain the successive approximations:

$$u_1(x,y,t) = u_0(x,y,t) -$$

$$-{}_0I_t^{(\alpha)}\{L_\xi^{(\alpha)}u_0(x,y,\xi) - D^\alpha[L_{xx}^{(2\alpha)}u_0(x,y,\xi) + L_{yy}^{(2\alpha)}u_0(x,y,\xi)]\} =$$

$$= E_\alpha(x^\alpha)E_\alpha(y^\alpha)\left[1 + \frac{2D^\alpha t^\alpha}{\Gamma(1+\alpha)}\right] \tag{11a}$$

$$u_2(x,y,t) = u_1(x,y,t) -$$

$$-{}_0I_t^{(\alpha)}\{L_\xi^{(\alpha)}u_1(x,y,\xi) - D^\alpha[L_{xx}^{(2\alpha)}u_1(x,y,\xi) + L_{yy}^{(2\alpha)}u_1(x,y,\xi)]\} =$$

$$= E_\alpha(x^\alpha)E_\alpha(y^\alpha)\left[1 + \frac{2D^\alpha t^\alpha}{\Gamma(1+\alpha)} + \frac{2^2 D^{2\alpha} t^{2\alpha}}{\Gamma(1+2\alpha)}\right] \tag{11b}$$

$$u_3(x,y,t) = u_2(x,y,t) -$$

$$-{}_0I_t^{(\alpha)}\{L_\xi^{(\alpha)}u_2(x,y,\xi) - D^\alpha[L_{xx}^{(2\alpha)}u_2(x,y,\xi) + L_{yy}^{(2\alpha)}u_2(x,y,\xi)]\} =$$

$$= E_\alpha(x^\alpha)E_\alpha(y^\alpha)\left[1 + \frac{2D^\alpha t^\alpha}{\Gamma(1+\alpha)} + \frac{2^2 D^{2\alpha} t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{2^3 D^{3\alpha} t^{3\alpha}}{\Gamma(1+3\alpha)}\right] \tag{11c}$$

$$\cdots\cdots$$

$$u_n(x,y,t) =$$

$$= E_\alpha(x^\alpha)E_\alpha(y^\alpha)\left[1 + \frac{2D^\alpha t^\alpha}{\Gamma(1+\alpha)} + \frac{2^2 D^{2\alpha} t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{2^3 D^{3\alpha} t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots + \frac{2^n D^{n\alpha} t^{n\alpha}}{\Gamma(1+n\alpha)}\right] \tag{11d}$$

Therefore, we estimate the non-differentiable solution for the 2-D diffusion problem (1) in fractal heat transfer:

$$u(x,y,t) = E_\alpha(x^\alpha)E_\alpha(y^\alpha)E_\alpha(2D^\alpha t^\alpha) \tag{12}$$

and its graph is illustrated in fig. 1 when $\alpha = \ln 2/\ln 3$, $D = 1$, and $y = 0$.

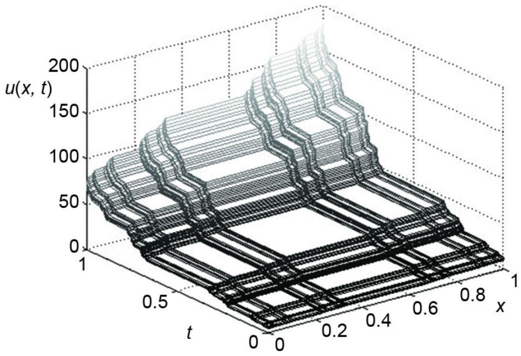

Figure 1. The non-differentiable solution for the 2-D diffusion problem in fractal heat transfer when α = ln2/ln3, D = 1, and y = 0

Conclusions

The LFVIM is a powerful tool which is capable of handling linear partial differential equations within the LFD. Using the presented technology, we have successfully solved the diffusion problem in fractal heat transfer. With help of the LFD operators, the result shows that a correction functional can be easily constructed by a general Lagrange multiplier, and this multiplier can be optimally identified by local fractional variational theory. The application of restricted variations in correction functional makes it much easier to determine the general multiplier.

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Nomenclature

t – time, [s]
$u(x, y, t)$ – concentration, [–]
x, y – space co-ordinates, [m]

Greek symbol

α – time fractal dimensional order, [–]

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