LAPLACE VARIATIONAL ITERATION METHOD FOR THE TWO-DIMENSIONAL DIFFUSION EQUATION IN HOMOGENEOUS MATERIALS

by

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In this paper, we suggest the local fractional Laplace variational iteration method to deal with the two-dimensional diffusion in homogeneous materials. The operator is considered in local fractional sense. The obtained solution shows the non-differentiable behavior of homogeneous materials with fractal characteristics.

Key words: *local fractional Laplace variational iteration method, diffusion equation, non-differentiable solution, local fractional derivative*

Introduction

Diffusion phenomenon [1] is one of basic characteristics of a wide variety of matters in nature. With help of non-local and local operators, the mathematical model of diffusion problems has been presented. For example, the fractional diffusion via Riemann-Liouville operator was suggested by Hilfer [2] and its generalized version was discussed in Sandev *et al.* [3]. Huang and Liu [4] presented the fractional diffusion via Caputo derivatives as well. The sub-diffusion with via Riemann-Liouville and Caputo derivatives was discussed in [5-7]. Tadjeran *et al.* [8] presented the super-diffusion via Riemann-Liouville and Caputo derivatives.

In real materials, there are fractal characteristics of surfaces of materials [9-11]. The non-local fractional operator is not adopted to describe the behaviors. The new model for the diffusion problem was presented [12-14]. The local fractional Laplace variational iteration method (LFVIM) was structured in [15] and developed to deal with non-homogeneous heat [16], vehicular traffic flow [17] and others [18]. Aim of this article is to use the method to find the solution for the 2-D diffusion problem in the homogeneous materials [16]:

$$\frac{\partial^{\kappa}\varphi(x,y,t)}{\partial\tau^{\kappa}} = \omega^{2\kappa}\nabla^{2\kappa}\varphi(x,y,\tau), \quad \kappa\in\left(0,1\right] \tag{1}$$

where the local fractional Laplace operator denotes [1, 19, 20]:

$$\nabla^{2\kappa} = \frac{\partial^{2\kappa}}{\partial x^{2\kappa}} + \frac{\partial^{2\kappa}}{\partial y^{2\kappa}} \tag{2}$$

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Preliminaries

The LFD of $\Omega(\eta)$ of order κ at $\eta=\eta_0$ is defined as [12-20]:

$$D_\eta^\kappa \Omega(\eta_0)=\Omega^{(\kappa)}(\eta_0)=\lim_{\eta\to\eta_0}\frac{\Delta^\kappa[\Omega(\eta)-\Omega(\eta_0)]}{(\eta-\eta_0)^\kappa} \tag{3}$$

where $\Delta^\kappa[\Omega(\eta)-\Omega(\eta_0)]\cong\Gamma(\kappa+1)[\Omega(\eta)-\Omega(\eta_0)]$.

The local fractional integral operator (LFI) of $\Omega(\eta)$ of order κ in the interval (ς,ζ) is defined as [12, 13, 15-18]:

$${}_\varsigma I_\xi^{(\kappa)}\Omega(\eta)=\frac{1}{\Gamma(1+\kappa)}\int_\varsigma^\xi \Omega(\eta)(\mathrm{d}\mu)^\kappa=\frac{1}{\Gamma(1+\kappa)}\lim_{\Delta\mu\to 0}\sum_{j=0}^{N-1}\Omega(\eta)(\Delta\mu)^\kappa \tag{4}$$

The local fractional Laplace transform (LFLT) of $\Omega(\eta)$ is defined as [15-18]:

$$L_\kappa\Omega(\eta)=\Omega_s^{L,\kappa}(s)=\frac{1}{\Gamma(1+\kappa)}\int_0^\infty E_\kappa(-s^\kappa\mu^\kappa)\Omega(\mu)(\mathrm{d}\mu)^\kappa,\quad 0<\kappa\le 1 \tag{5}$$

where the latter integral converges and $s\in R$.

Its inverse formula is defined as [15-18]:

$$L_\kappa^{-1}[\Omega_s^{L,\kappa}(s)]=\Omega(\mu)=\frac{1}{(2\pi)^\kappa}\int_{\beta-i\omega}^{\beta+i\omega}E_\kappa(s^\kappa\mu^\kappa)\Omega_s^{L,\kappa}(s)(\mathrm{d}s)^\kappa,\quad 0<\kappa\le 1 \tag{6}$$

where $s=\beta+i\omega$ and $\mathrm{Re}(s)=\beta>0$.

The local fractional convolution of two functions is given as [15-18]:

$$\Omega_1(\eta)\,\Omega_2(\eta)=\frac{1}{\Gamma(1+\kappa)}\int_0^\infty \Omega_1(\tau)\,\Omega_2(\eta-\tau)(\mathrm{d}\tau)^\kappa \tag{7}$$

and we have:

$$L_\kappa\{\Omega_1(\eta)\,\Omega_2(\eta)\}=\tilde{L}_\kappa\{\Omega_1(\eta)\}L_\kappa\{\Omega_2(\eta)\} \tag{8}$$

Analysis of the LFLVIM

We write eq. (1) in the local fractional operator:

$$T_\kappa u-R_\kappa u=0 \tag{9}$$

where $T_\kappa=\partial^\kappa/\partial\tau^\kappa$, and $R_\kappa=\omega^{2\kappa}(\partial^{2\kappa}/\partial x^{2\kappa}+\partial^{2\kappa}/\partial y^{2\kappa})$.

The local fractional functional formula can be structured as:

$$u_{n+1}(x,y,\tau)=u_n(x,y,\tau)+{}_0I_\tau^{(\kappa)}\left\{\frac{\lambda(\eta-\tau)}{\Gamma(1+\kappa)}[T_\kappa u_n(x,y,\tau)-R_\kappa u_n(x,y,\tau)]\right\} \tag{10}$$

Adopting eq. (5), eq. (10) becomes:

$$L_\kappa\{u_{n+1}(x,y,\tau)\}=L_\kappa\{u_n(x,y,\tau)\}+L_\kappa\left\{\frac{\lambda(\eta-\tau)}{\Gamma(1+\kappa)}\right\}L_\kappa\{[T_\kappa u_n(x,y,\tau)-R_\kappa u_n(x,y,\tau)]\} \tag{11}$$

Taking local fractional variation [12], eq. (11) can be written as:

$$\delta^{\kappa}\{u_{n+1}(x,y,\tau)\} = \\ = \delta^{\kappa}\{L_{\kappa}\{u_n(x,y,\tau)\}\} + \delta^{\kappa}\left\{L_{\kappa}\frac{\lambda(\eta-\tau)}{\Gamma(1+\kappa)}\right\}L_{\kappa}\{[T_{\kappa}u_n(x,y,\tau) - R_{\kappa}u_n(x,y,\tau)]\} \tag{12}$$

Hence, we have:

$$\delta^{\kappa}\{L_{\kappa}\{u_n(x,y,\tau)\}\} = \delta^{\kappa}\{L_{\kappa}\{u_n(x,y,\tau)\}\} + \delta^{\kappa}\left\{L_{\kappa}\left\{\frac{\lambda(\eta-\tau)}{\Gamma(1+\kappa)}\right\}L_{\kappa}[T_{\kappa}u_n(x,y,\tau)]\right\} = 0 \tag{13}$$

From eq. (13), we get:

$$\delta^{\kappa}\{L_{\kappa}\{\mathrm{T}_{\kappa}(x,y,\tau)\}\} = \delta^{\kappa}\{s^{\kappa}L_{\kappa}\{u_n(x,y,\tau)\} - s^{\kappa}u_n(x,y,0)\} = s^{\kappa}\delta^{\kappa}L_{\kappa}\{u_n(x,y,\tau)\} \tag{14}$$

such that:

$$1 + L_{\kappa}\left\{\frac{\lambda(\tau)}{\Gamma(1+\nu)}\right\}s^{\kappa} = 0 \tag{15}$$

From eq. (15) we have:

$$L_{\kappa}\left\{\frac{\lambda(\tau)}{\Gamma(1+\nu)}\right\} = -\frac{1}{s^{\kappa}} \tag{16}$$

such that local fractional iteration algorithm can be written as:

$$L_{\kappa}\{u_{n+1}(x,y,\tau)\} = L_{\kappa}\{u_n(x,y,\tau)\} - \frac{1}{s^{\kappa}}L_{\kappa}\{[T_{\kappa}u_n(x,y,\tau) - R_{\kappa}u_n(x,y,\tau)]\} \tag{17}$$

subject to the initial value condition:

$$L_{\kappa}\{u_0(x,y,\tau)\} = L_{\kappa}\{u(x,y,0)\} \tag{18}$$

Therefore, the solution in non-differentiable series is written as:

$$L_{\kappa}\{u(x,y,\tau)\} = \lim_{n\to\infty} L_{\kappa}\{u_n(x,y,\tau)\} \tag{19}$$

Taking the inversion LFLT, one obtain:

$$u(x,y,\tau) = L_{\kappa}^{-1}\{\lim_{n\to\infty} L_{\kappa}\{u_n(x,y,\tau)\}\} \tag{20}$$

2-D diffusion problem in homogeneous materials

We now consider 2-D diffusion problem in homogeneous materials where:

$$\omega^{2\kappa} = \frac{1}{2} \tag{21}$$

and the initial conditions are given as:

$$u(x,y,0)=E_\kappa(x^\kappa)E_\kappa(y^\kappa) \tag{22}$$

$$u(0,y,\tau)=E_\kappa(y^\kappa)E_\kappa(\tau^\kappa) \tag{23}$$

$$u(x,0,t)=E_\kappa(x^\kappa)E_\kappa(\tau^\kappa) \tag{24}$$

In view of eqs. (22)-(24), we have the initial value given as:

$$L_\kappa\{u(x,y,\tau)\}=\frac{1}{s^\kappa}E_\kappa(x^\kappa)E_\kappa(y^\kappa) \tag{25}$$

Making use of eq. (17), the approximations read as:

$$L_\kappa\{u_1(x,y,\tau)\}=L_\kappa\{u_0(x,y,\tau)\}-\frac{1}{s^\kappa}L_\kappa\{[T_\kappa u_0(x,y,\tau)-R_\kappa u_0(x,y,\tau)]\}=$$

$$=E_\kappa(x^\kappa)E_\kappa(y^\kappa)\left(\frac{1}{s^\kappa}+\frac{1}{s^{2\kappa}}\right) \tag{26}$$

$$L_\kappa\{u_2(x,y,\tau)\}=L_\kappa\{u_1(x,y,\tau)\}-\frac{1}{s^\kappa}L_\kappa\{[T_\kappa u_1(x,y,\tau)-R_\kappa u_1(x,y,\tau)]\}=$$

$$=E_\kappa(x^\kappa)E_\kappa(y^\kappa)\left(\frac{1}{s^\kappa}+\frac{1}{s^{2\kappa}}+\frac{1}{s^{3\kappa}}\right) \tag{27}$$

$$L_\kappa\{u_3(x,y,\tau)\}=L_\kappa\{u_2(x,y,\tau)\}-\frac{1}{s^\kappa}L_\kappa\{[T_\kappa u_2(x,y,\tau)-R_\kappa u_2(x,y,\tau)]\}=$$

$$=E_\kappa(x^\kappa)E_\kappa(y^\kappa)\left(\frac{1}{s^\kappa}+\frac{1}{s^{2\kappa}}+\frac{1}{s^{3\kappa}}+\frac{1}{s^{4\kappa}}\right) \tag{28}$$

$$L_\kappa\{u_4(x,y,\tau)\}=L_\kappa\{u_3(x,y,\tau)\}-\frac{1}{s^\kappa}L_\kappa\{[T_\kappa u_3(x,y,\tau)-R_\kappa u_3(x,y,\tau)]\}=$$

$$=E_\kappa(x^\kappa)E_\kappa(y^\kappa)\left(\frac{1}{s^\kappa}+\frac{1}{s^{2\kappa}}+\frac{1}{s^{3\kappa}}+\frac{1}{s^{4\kappa}}+\frac{1}{s^{5\kappa}}\right) \tag{29}$$

and so on.

Consequently, the non-differentiable solution in the local fractional series is presented as:

$$u(x,y,\tau)=L_\kappa^{-1}\left\{\lim_{n\to\infty}L_\kappa\{u_n(x,y,\tau)\}\right\}=\lim_{n\to\infty}\left[E_\kappa(x^\kappa)E_\kappa(y^\kappa)\sum_{k=0}^{n}\frac{1}{s^{(k+1)\kappa}}\right]=$$

$$=E_\kappa(x^\kappa)E_\kappa(y^\kappa)E_\kappa(\tau^\kappa) \tag{30}$$

and its graph is shown in fig. 1 when $\kappa=\ln 2/\ln 3$.

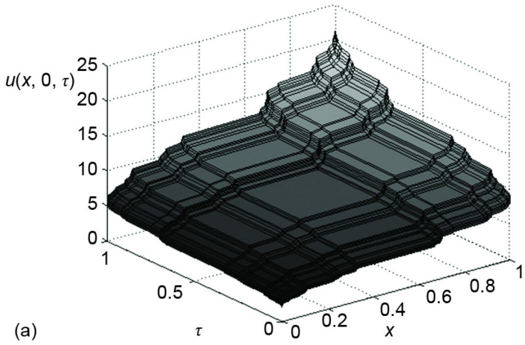

Figure 1(a) The non-differentiable solution eq. (30) when $y = 0$

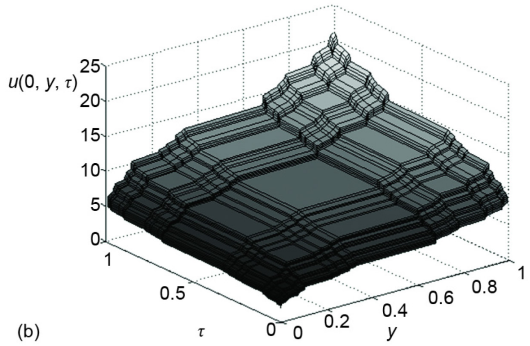

Figure 1(b) The non-differentiable solution eq. (30) when $x = 0$

Conclusions

In this work, we have discussed the 2-D diffusion problem in homogeneous materials. The LFLVIM is successfully applied to find the closed solution with non-differentiable condition. The obtained result shows the technology is simpler, easier, effective and explicit.

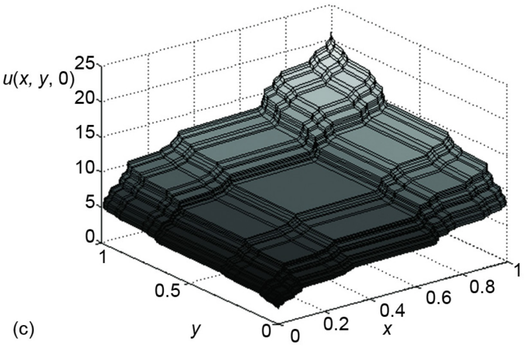

Figure 1(c) The non-differentiable solution eq. (30) when $t = 0$

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Nomenclature

x, y	– space co-ordinates, [m]
$u\,(x, y, \tau)$	– the concentration distribution, [–]
$L_\kappa[\Omega(\eta)]$	– LFLT, [–]
$L_\kappa^{-1}[\Omega_s^{L,\kappa}(s)]$	– inverse formula of LFLT, [–]

Greek symbols

κ	– time fractal dimensional order, [–]
τ	– time, [s]

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