ON TWO EXACT SOLUTIONS OF TIME FRACTIONAL HEAT EQUATIONS USING DIFFERENT TRANSFORMS

by

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Original scientific paper
DOI: 10.2298/TSCI15S1S43I

Two solutions of time fractional differential equations are illustrated. The first one converges to functional space in term of Weyl transform in $L^2(\mathbb{R})$, while the second solution approaches to the Fox function with respect to time, by using the Fourier and Laplace-Mellin transforms. The fractional calculus is taken in the sense of the Riemann-Liouville fractional differential operator.

Key words: *fractional calculus, fractional diffusion equation, fractional heat equation*

Introduction

In the last few decades, the class of fractional differential equations has been contemplated to be successful patterns of real life phenomenon. One of the essential applications of the fractional calculus is formed by the intermediate physical process. A very significant type is the fractional diffusion and wave equations. It has been found that the diversity of the universal electromagnetic, acoustic, mechanical, and responses can be formulated precisely applying fractional diffusion-wave equations [1-6].

Various considerations of the fractional diffusion equations have been imposed in different fractional operators such as the Riemann-Liouville, Caputo, and Rize fractional differential operators [7, 8]. Furthermore, the authors studied a maximal solution of the time-space fractional heat equation in a complex domain. The fractional time is considered in the sense of the Riemann-Liouville operator, while the fractional space is introduced in the Srivastava-Owa operator for complex variables [9].

Transform is a considerable mechanism to deal with mathematical problems by finding exact and approximate solutions. numerous beneficial transforms for solving unlike problems adapted in wide literatures, such as the Laplace transform, the Fourier transform, wavelet transformation, the Bucklund transformation, the local fractional integral transforms, the integral transform, the fractional complex transforms, the Hankel integral transforms, and matrix transform Miura type [10-15].

Recently, the fractional heat equation is studied by many authors. Anwar *et al*. [16] imposed the double Laplace transform of the partial fractional integrals and derivatives to get a solution of partial differential equations in the sense of the Caputo operator. Yang and Baleanu [17] processed the heat equation of fractional order by using the variations iterative

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method. Chen *et al.* [18] established the existence and uniqueness of weak solution for a class of fractional heat equation.

In this study, the two solutions of time fractional differential equations are illustrated. The first one converges to functional space in term of Weyl transform in $L^2(\mathsf{R})$ while the second solution approaches to the Fox function with respect to time, by using the Fourier and Laplace-Mellin transforms. The fractional calculus based on the the Riemann-Liouville fractional differential operator.

Mathematical setting

Let $\sigma \in L^2(\mathsf{C})$. The Weyl transform $\mathsf{þ}_\sigma : L^2(\mathsf{R}) \to L^2(\mathsf{R})$ is defined by [19]:

$$(\mathsf{þ}_\sigma \phi, \psi)_{L^2(\mathsf{R})} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(x, y) \mu(\varphi, \psi)(x, y) \mathrm{d}x \mathrm{d}y, \quad \varphi, \psi \in L^2(\mathsf{R})$$

where μ is the Wigner transform of φ and ψ. Consider the set of all functions φ by $\mathsf{F}(\mathsf{C})$ such that:

$$\int_{\mathsf{C}} |\varphi(z)|^2 \psi(z) \mathrm{d}z < \infty$$

where dz refers to the Lebesgue measure on the complex plan C, and:

$$\psi(z) = \pi^{-1} \exp(-|z|^2), \quad z \in \mathsf{C}$$

Let θ be a measurable function on C. The Bargmann transform $B\theta$ is defined by:

$$(B\theta)(z) = \pi^{-1/4} \int_{-\infty}^{\infty} \theta(\chi) \exp \frac{\sqrt{2}\pi \chi z - (\chi^2 + z^2)}{2} \mathrm{d}\chi < \infty, \quad z \in \mathsf{C}$$

Lemma 1. Let $\varphi \in \mathsf{F}(\mathsf{C})$. Then:

$$(B \mathsf{þ}_{\sigma(\eta)} B^{-1} \varphi)(z) = g(\eta) \varphi(\eta z), \quad z \in \mathsf{C}$$

Let $\hbar$ be a non negative differential function on $(0, \infty)$ satisfying the condition:

$$0 < \hbar(t) < 1, \quad t \in (0, \infty)$$

and

$$\lim_{t \to 0} \hbar(t) = 1$$

Let $f \in L^2(\mathsf{R})$. Our aim is to determine the solution of the fractional heat equation of the form:

$$\frac{\partial^\wp T}{\partial t^\wp}(\chi, t) = \sum_{n=1}^{k} w_n(\chi) T(\chi, t) \tag{1}$$

for each $k \in \mathsf{N}$, $\chi \in \mathsf{R}, t > 0$, $\wp \in (0,1)$ such that:

$$T(., t) \to f \in L^2(\mathsf{R}), \quad \text{as} \quad t \to 0 \tag{2}$$

where $\partial^\wp$ denoted the Riemann-Liouville fractional differential operators [1]:

$$\frac{\partial_a^\wp}{\partial t^\wp} \upsilon(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{(t-\tau)^{-\wp}}{\Gamma(1-\wp)} \upsilon(\tau) \mathrm{d}\tau$$

and for $a = 0$ we consider the form $\partial^{\wp}/\partial t^{\wp}$.

We need the following results:

Lemma 2. [20] Let one of the following assumptions be achieved:

$$\mu \in \eth([0, K]) \text{ and } \nu \in \eth^{\beta}([0, K]), \quad \wp < \beta \le 1$$
$$\nu \in \eth([0, K]) \text{ and } \mu \in \eth^{\beta}([0, K]), \quad \wp < \beta \le 1$$
$$\mu \in \eth^{\beta}([0, K]) \text{ and } \nu \in \eth^{\delta}([0, K]), \quad \wp < \beta \le \beta + \delta, \quad \beta, \quad \delta \in (0,1)$$

where $\eth^{\gamma}([0, K]) = \{\ell : [0, K] \to \mathsf{R} / \,|\, \ell(s) - \ell(s - j)\,| = O(j^{\gamma}) \text{ uniformly for } 0 < \tau - j < s \le K\}$.

Then

$$\partial_s^{\wp}(\ell b)(s) = \ell(s)\partial_s^{\wp} b(s) + b(s)\partial_s^{\wp}\ell(s) + R_{\wp}(s)$$

where

$$R_{\wp}(s) := -\frac{\wp}{\Gamma(1-\wp)}\int_0^s \frac{[\ell(\tau) - \ell(s)][b(\tau) - b(s)]}{(s-\tau)^{\wp+1}}\mathrm{d}\tau - \frac{\ell(s)b(s)}{\Gamma(1-\wp)s^{-\wp}}$$

point-wise.

Space function solution

This section deals with the existence of solutions of the initial value problem (1)-(2). Note that the uniqueness follow by the general theory for the heat equations.

Theorem 1. The solution $T(\chi, t)$ of the initial value problem (1)-(2) is presented by:

$$T(\chi, t) = (\mathsf{þ}_{\sigma(\hbar)} f)(\chi), \quad \chi \in \mathsf{R}, \quad t > 0$$

Proof. Define a function T on the set:

$$S := \{(\zeta, \tau) : \zeta \in \mathsf{C}, \quad \tau > 0\}$$

formulated by:

$$\mathrm{T}(z, t) = (BT)(\zeta, \tau), \quad (\zeta, \tau) \in S$$

and let φ be the function on C introduced by:

$$\varphi(\zeta) = (Bf)(\zeta), \quad \zeta \in \mathsf{C}$$

By Lemma 1, we receive that for each $\zeta \in \mathsf{C}$ and $\tau > 0$:

$$\mathrm{T}(\zeta, \tau) = (B\mathsf{þ}_{\sigma(\hbar)} B^{-1}\varphi)(\zeta) = g[\hbar(\tau)]\varphi[\hbar(\tau)\zeta]$$

Consequently, by Lemma 2, we obtain:

$$\frac{\partial^{\wp}\mathrm{T}}{\partial t^{\wp}}(\zeta, \tau) = g[\hbar(\tau)]\frac{\partial^{\wp}\varphi}{\partial \tau^{\wp}}[\hbar(\tau)\zeta] + \frac{\partial^{\wp} g}{\partial \tau^{\wp}}[\hbar(\tau)]\varphi[\hbar(\tau)\zeta] + \frac{\partial^{\wp} R_{\wp}}{\partial \tau^{\wp}}(\tau)$$

By the definition of $\hbar$ yields that $R_{\wp}(\tau) = 0$. Thus, we obtain:

$$\frac{\partial^{\wp}\mathrm{T}}{\partial t^{\wp}}(\zeta,\tau)=g[\hbar(\tau)]\frac{\partial^{\wp}\varphi}{\partial\tau^{\wp}}[\hbar(\tau)\zeta]+\frac{\partial^{\wp}g}{\partial\tau^{\wp}}[\hbar(\tau)]\varphi[\hbar(\tau)\zeta]=$$

$$=g[\hbar(\tau)]\varphi[\hbar(\tau)\zeta]\frac{\frac{\partial^{\wp}\varphi}{\partial\tau^{\wp}}[\hbar(\tau)\zeta]}{\varphi[\hbar(\tau)\zeta]}+\frac{\frac{\partial^{\wp}g}{\partial\tau^{\wp}}[\hbar(\tau)]}{g[\hbar(\tau)]}g[\hbar(\tau)]\varphi[\hbar(\tau)\zeta]= \tag{3}$$

$$:=w_1(\zeta)\mathrm{T}(\zeta)+w_2(\zeta,\tau)\mathrm{T}(\zeta,\tau)$$

where $\lim\hbar(\tau)\to 1.$ Thus, again in view of Lemma 2, we have:

$$\frac{\partial^{\wp}(BT)}{\partial\tau^{\wp}}(\zeta,\tau)=w_1(\zeta)(BT)(\zeta,\tau)+w_2(\zeta)(BT)(\zeta,\tau) \tag{4}$$

By taking the inverse Bargmann transform and applying Lemma 2, we attain to:

$$\frac{\partial^{\wp}(T)}{\partial\tau^{\wp}}(\chi,\tau)=\sum_{n=1}^{k}w_n(\chi)T(\chi,\tau),\quad k=3 \tag{5}$$

where

$$B^{-1}\frac{\partial^{\wp}(BT)}{\partial t^{\wp}}(\chi,\tau)=\frac{\partial^{\wp}T}{\partial\tau^{\wp}}(\chi,\tau)-BT\frac{\partial^{\wp}B^{-1}}{\partial\tau^{\wp}}(\chi,t)-R_{\wp}$$

and from the condition $\lim_{t=0}\hbar(\tau)=1$, we conclude that $R_{\wp}(\chi,\tau)=0.$ Thus:

$$w_3(\chi):=B\frac{\partial^{\wp}B^{-1}}{\partial\tau^{\wp}}$$

We proceed to prove that:

$$T(.,\tau)\to f\in L^2(\mathrm{R}),\quad \tau\to 0$$

In virtue of the Lebesgue's dominated convergence theorem and spectral theorem, we conclude that:

$$\|\mathrm{þ}_{\sigma}(\hbar)f-f\|^2_{L^2(\mathrm{R})}\to 0,\quad \text{as}\quad \tau\to 0$$

This completes the proof.

Theorem 2. For each $t > 0$:

$$\|\ T(.,t)\ \|_{L^2(\mathrm{R})}\le C[\hbar(t)]\|\ f\ \|_{L^2(\mathrm{R})}$$

where $C[\hbar(t)]$ is a positive function on $(0,\infty)$.

Proof. It is clear that:

$$\|T(.,t)\|_{L^2(\mathrm{R})}=\|\mathrm{þ}_{\sigma}(\hbar)f\|_{L^2(\mathrm{R})}$$

Therefore, by the Cauchy-Schwartz inequality yields:

$$\|\mathrm{þ}_{\sigma}(\hbar)f\|_{L^2(\mathrm{R})}\le\|\sigma(\hbar)\|_{L^2(\mathrm{R})}\|f\|_{L^2(\mathrm{R})}$$

Assume that $C[\hbar(t)]:=\|\sigma(\hbar)\|_{L^2(\mathrm{R})}$ consequently, we obtain the desired assertion.

Time solution

In this section, we process another solution for the initial value problem (1)-(2) in term of time.

Theorem 3. For each $t > 0$:

$$T(x,t) : \frac{1}{t^{\wp}}, \quad \wp \in (0,1)$$

Proof. By employing the fractional integral operator:

$$I_t^{\wp}\upsilon(t) = \int \frac{(t-\tau)^{\wp-1}}{\Gamma(\wp)}\upsilon(\tau)\mathrm{d}\tau$$

on eq. (1), we have:

$$T(\chi,t) = f(\chi) + \sum_{n=1}^{k} w_n(\chi) I_t^{\wp} T(\chi,t) \tag{6}$$

The Fourier transform of the function T is computed by the integral:

$$T(x,t) = \int_{-\infty}^{\infty} T(\chi,t)\exp(ix\chi)\mathrm{d}\chi$$

Thus eq. (6) in the Fourier domain yields:

$$T(x,t) = 1 + I_t^{\wp} W_n(x) T(x,t), \quad n \in \mathrm{N} \tag{7}$$

where $W_n(x)$ is the Fourier transform of the coefficients. A Laplace transform of eq. (7) implies that:

$$T(x,\hat{\mathrm{E}}) = \frac{\hat{\mathrm{E}}^{\wp-1}}{\hat{\mathrm{E}}^{\wp} - W_n(x)} \tag{8}$$

To reduce x in term of time, we utilize the mean square displacement, which is given by:

$$<\chi^2(t)> = \int T(\chi,t)\chi^2 \mathrm{d}\chi = -cT(x,t)|_{x=0}$$

where $c \geq 0$. This leads to:

$$<\chi^2(t)> : t^{\wp}$$

By applying the Mellin transforms:

$$\lambda(\varsigma) = M[\lambda(\tau),\varsigma] = \int_0^{\infty} t^{\varsigma-1}\lambda(\tau)\mathrm{d}\tau$$

and utilizing the relation:

$$M[\lambda(\tau),\varsigma] = \frac{1}{\Gamma(1-\varsigma)} M\{Ł[\lambda(\tau),\hat{\mathrm{E}}], 1-\varsigma\}$$

where Ł is the Laplace transform, we obtain:

$$T(x,\varsigma)=\frac{1}{\Gamma(1-\varsigma)}\int_0^\infty \frac{\hat{\mathsf{E}}^{\wp-\varsigma-1}}{W_n+\hat{\mathsf{E}}^{\wp-1}}\mathrm{d}\hat{\mathsf{E}}$$

where the assertion is the inverse Laplace transform of eq. (8). Consequently, we receive that:

$$T(x,\varsigma)=\frac{1}{\wp\Gamma(1-\varsigma)}W_n^{-\varsigma/\wp}\beta[(\varsigma/\wp),(1-\varsigma/\wp)] \tag{9}$$

where β denoted the beta function. The inversion of eq. (9) into the time domain implies:

$$T(x,t)=\frac{1}{\wp}\mathsf{H}_{1,2}^{1,1}\left[W_n^{1/\wp}t\,|_{(0,1/\wp),\,(0,1)}^{(0,1/\wp)}\right]$$

where H is the Fox-Wright function:

$${}_q\mathsf{H}_p\left[\eta\,\Bigg|\begin{matrix}(b_1,\beta_1),...,(b_q,\beta_q)\\ \\ (a_1,\alpha_1),...,(a_p,\alpha_p)\end{matrix}\right]={}_q\mathsf{H}_p[\eta;(b_j,\beta_j)_{1,q};\;(a_j,\alpha_j)_{1,p}]$$

$$:=\sum_{n=0}^{\infty}\frac{\Gamma(b_1+n\beta_1)\ldots\Gamma(b_q+n\beta_q)}{\Gamma(a_1+n\alpha_1)\ldots\Gamma(a_q+n\alpha_p)}\frac{\eta^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma(b_j+n\beta_j)}{\prod_{j=1}^{p}\Gamma(a_j+n\alpha_j)}\frac{\eta^n}{n!}$$

with $a_j,b_j\in\mathsf{R}$, $\beta_j>0$ for all $j=1,...,q$, $\alpha_j>0$ for all $j=1,...,p$ and $1+\sum_{j=1}^{p}\alpha_j-$ $-\sum_{j=1}^{q}\beta_j\geq 0$ for $|\eta|<1$. But $\mathsf{H}_{1,2}^{1,1}$ can be expanded in power series:

$$T(x,t)=\frac{1}{\wp}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(2n\wp)}(W_nt)^{2n\wp}$$

Hence, the solution is monotonic decreasing function with the asymptotic behavior at $t\to\infty$:

$$T(x,t):\frac{1}{t^{\wp}},\quad t>0$$

This completes the proof.

Conclusions

We conclude that the solutions of the fractional differential heat equation, under some special conditions, can be converted into two formulas. The first one depends on the function space, which is given by the initial condition. While, the second one translated into time domain by using the concept of the asymptotic behavior, when $t\to\infty$. We utilized different transoms in both cases. On the first case, we employed the Weyl transform in term of the Wigner transform and the Bargmann transform (see Theorem 3). The second case deled with the Fourier transform together with the Laplace transform as well as the Mellin transform. This viewed that the theory of transforms is very active to convert solutions.

Acknowledgments

This research is supported by Project No.: RG312-14AFR from the University of Malaya.

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Paper submitted: October 10, 2014
Paper revised: January 21, 2015
Paper accepted: February 2, 2015