ADOMIAN DECOMPOSITION METHOD FOR THREE-DIMENSIONAL DIFFUSION MODEL IN FRACTAL HEAT TRANSFER INVOLVING LOCAL FRACTIONAL DERIVATIVES

by

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The non-differentiable analytical solution of the 3-D diffusion equation in fractal heat transfer is investigated in this article. The Adomian decomposition method is considered in the local fractional operator sense. The obtained result is given to show the sample and efficient features of the presented technique to implement fractal heat transfer problems.

Key words: *Adomian decomposition method, diffusion equation, fractal heat transfer, local fractional derivative*

Introduction

The theory of local fractional calculus attracts researchers from mathematical physics and engineering applications [1-8]. This interest spans the works of diffusion phenomena with non-differentiability [9-11]. The 3-D diffusion model in fractal heat transfer involving local fractional derivatives (LFD) was presented as [2, 8]:

$$\eta^{\alpha}\nabla^{2\alpha}\Phi(x,y,z,\tau)=\frac{\partial^{\alpha}\Phi(x,y,z,\tau)}{\partial\tau^{\alpha}} \tag{1}$$

subject to the initial and boundary conditions:

$$\Phi(x,y,z,0)=f(x,y,z) \tag{2a}$$

$$\Phi(0,y,z,\tau)=\Phi(a,y,z,\tau)=g_1(y,z,\tau) \tag{2b}$$

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$$\Phi(x,0,z,\tau)=\Phi(x,b,z,\tau)=g_2(x,z,\tau) \tag{2c}$$

$$\Phi(x,y,0,t)=\Phi(x,y,c,t)=g_3(x,y,t) \tag{2d}$$

where the local fractional Laplace operator is defined as [1, 2, 4-8]:

$$\nabla^{2\alpha}=\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}+\frac{\partial^{2\alpha}}{\partial z^{2\alpha}} \tag{3}$$

η^β is a non-differentiable diffusion coefficient, and $\Phi(x,y,z,\tau)$ is satisfied with the non-differentiable concentration distribution [2, 9]. Recently, the authors [10] suggested the local fractional Adomian decomposition method (LFADM) to consider 1-D diffusion equation on Cantor time-space. Based on it, Yan *et al.* considered the Laplace equation within the LFD [11]. Baleanu *et al.* developed non-differential solution to wave equation on Cantor sets within the LFD [12]. The main target of this manuscript to utilize the method to implement the 3-D diffusion model in fractal heat transfer.

3-D diffusion model in fractal heat transfer

We first rewrite the problem (1) in the local fractional operator form:

$$L_\tau^{(\alpha)}\Phi(x,y,z,\tau)=\eta^\alpha[L_{xx}^{(2\alpha)}\Phi(x,y,z,\tau)+L_{yy}^{(2\alpha)}\Phi(x,y,z,\tau)+L_{zz}^{(2\alpha)}\Phi(x,y,z,\tau)] \tag{4}$$

where the local fractional differential operators (see A1 of the Appendix) $L_t^{(\alpha)}$, $L_{xx}^{(2\alpha)}$, $L_{yy}^{(2\alpha)}$, and $L_{zz}^{(2\alpha)}$ are defined by:

$$L_\tau^{(\alpha)}(.)=\frac{\partial^\alpha}{\partial\tau^\alpha}(.)\,,\quad L_{xx}^{(2\alpha)}(.)=\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}(.)\,,\quad L_{yy}^{(2\alpha)}(.)=\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}(.)\,,$$

$$L_{zz}^{(2\alpha)}(.)=\frac{\partial^{2\alpha}}{\partial z^{2\alpha}}(.) \tag{5a,b,c,d}$$

Adopting the inverse operator (see A2 of the Appendix) $L_\tau^{(-\alpha)}$ to both sides of (4) and using the initial condition leads to:

$$L_\tau^{(-\alpha)}L_\tau^{(\alpha)}\Phi(x,y,z,\tau)=$$

$$=\eta^\alpha L_\tau^{(-\alpha)}[\Phi_{xx}^{(2\alpha)}\Phi(x,y,z,\tau)+\Phi_{yy}^{(2\alpha)}\Phi(x,y,z,\tau)+L_{zz}^{(2\alpha)}\Phi(x,y,z,\tau)] \tag{6}$$

Hence, we get:

$$\Phi(x,y,z,\tau)=$$

$$=\eta^\alpha L_t^{(-\alpha)}[L_{xx}^{(2\alpha)}\Phi(x,y,z,\tau)+L_{yy}^{(2\alpha)}\Phi(x,y,z,\tau)+L_{zz}^{(2\alpha)}\Phi(x,y,z,\tau)]+\Phi(x,y,z,0) \tag{7}$$

According to the LFADM we decompose the unknown function $\Phi(x,y,z,\tau)$ as an infinite series:

$$\Phi(x,y,z,\tau)=\sum_{n=0}^{\infty}\Phi_n(x,y,z,\tau) \tag{8}$$

Substituting (8) into (7) yield:

$$\sum_{n=0}^{\infty}\Phi_n = \Phi(x,y,z,0) + \eta^{\alpha} L_{\tau}^{(-\alpha)}\left[L_{xx}^{(2\alpha)}\left(\sum_{n=0}^{\infty}\Phi_n\right) + L_{yy}^{(2\alpha)}\left(\sum_{n=0}^{\infty}\Phi_n\right) + L_{zz}^{(2\alpha)}\left(\sum_{n=0}^{\infty}\Phi_n\right)\right] \tag{9}$$

The components $\Phi_n(x,y,z,\tau)$, $n \geq 0$ can be completely determined by using the cursive relationship:

$$\Phi_0(x,y,z,\tau) = \Phi(x,y,z,0) \tag{10a}$$

$$\Phi_{n+1}(x,y,z,\tau) = \eta^{2\alpha} L_{\tau}^{(-\alpha)}[L_{xx}^{(2\alpha)}\Phi_n + L_{yy}^{(2\alpha)}\Phi_n + L_{zz}^{(2\alpha)}\Phi_n], \quad n \geq 0 \tag{10b}$$

Taking

$$\Phi(x,y,z,0) = \sin_{\alpha}(x^{\alpha})\cos_{\alpha}(y^{\alpha})\cos_{\alpha}(z^{\alpha}) \tag{11a}$$

$$\Phi(0,y,z,\tau) = \Phi(\pi,y,z,\tau) = 0 \tag{11b}$$

$$\Phi(x,0,z,\tau) = -\Phi(x,\pi,z,\tau) = 3E_{\alpha}[-(2\tau)^{\alpha}]\sin_{\alpha}(x^{\alpha})\cos_{\alpha}(z^{\alpha}) \tag{11c}$$

$$\Phi(x,y,0,\tau) = -\Phi(x,y,\pi,\tau) = 3E_{\alpha}[-(2\tau)^{\alpha}]\sin_{\alpha}(x^{\alpha})\cos_{\alpha}(y^{\alpha}) \tag{11d}$$

$$\eta = 0.2 \tag{11e}$$

we have:

$$\Phi_0(x,y,z,\tau) = \sin_{\alpha}(x^{\alpha})\cos_{\alpha}(y^{\alpha})\cos_{\alpha}(z^{\alpha}) \tag{12a}$$

$$\Phi_{n+1}(x,y,z,\tau) = 0.2^{\alpha} L_{\tau}^{(-\alpha)}[L_{xx}^{(2\alpha)}\Phi_n + L_{yy}^{(2\alpha)}\Phi_n + L_{zz}^{(2\alpha)}\Phi_n], \quad n \geq 0 \tag{12b}$$

Consequently, we obtain:

$$\Phi_0(x,y,z,\tau) = \sin_{\alpha}(x^{\alpha})\cos_{\alpha}(y^{\alpha})\cos_{\alpha}(z^{\alpha}) \tag{13}$$

$$\Phi_1(x,y,\tau) = 0.2^{\alpha} L_{\tau}^{(-\alpha)}[L_{xx}^{(2\alpha)}T_0 + L_{yy}^{(2\alpha)}T_0 + L_{zz}^{(2\alpha)}T_0] =$$

$$= -\frac{3(0.2\tau)^{\alpha}}{\Gamma(1+\alpha)}\sin_{\alpha}(x^{\alpha})\cos_{\alpha}(y^{\alpha})\cos_{\alpha}(z^{\alpha}) \tag{14}$$

$$\Phi_2(x,y,\tau) = 0.2^{\alpha} L_{\tau}^{(-\alpha)}[L_{xx}^{(2\alpha)}T_1 + L_{yy}^{(2\alpha)}T_1 + L_{zz}^{(2\alpha)}T_1] =$$

$$= \frac{3(0.2\tau)^{2\alpha}}{\Gamma(1+2\alpha)}\sin_{\alpha}(x^{\alpha})\cos_{\alpha}(y^{\alpha})\cos_{\alpha}(z^{\alpha}) \tag{15}$$

$$\Phi_3(x,y,\tau) = 0.2^{\alpha} L_{\tau}^{(-\alpha)}[L_{xx}^{(2\alpha)}T_2 + L_{yy}^{(2\alpha)}T_2 + L_{zz}^{(2\alpha)}T_2]$$

$$= -\frac{3(0.2\tau)^{3\alpha}}{\Gamma(1+3\alpha)}\sin_{\alpha}(x^{\alpha})\cos_{\alpha}(y^{\alpha})\cos_{\alpha}(z^{\alpha}) \tag{16}$$

and so on. The solution in a non-differentiable series form:

$$\Phi(x,y,z,\tau)=3\sin_\alpha(x^\alpha)\cos_\alpha(y^\alpha)\cos_\alpha(z^\alpha)\sum_{i=0}^{\infty}(-1)^i\frac{(0.2\tau)^{i\alpha}}{\Gamma(1+i\alpha)} \tag{17}$$

is readily obtained.

Therefore, the exact solution can be written as:

$$\Phi(x,y,z,\tau)=3E_\alpha[-(0.2\tau)^\alpha]\ \sin_\alpha(x^\alpha)\cos_\alpha(y^\alpha)\cos_\alpha(z^\alpha) \tag{18}$$

Figure 1 shows the exact solution of the 3-D diffusion model in fractal heat transfer when $\alpha=\ln 2/\ln 3$, z = 0, and $\tau=0$.

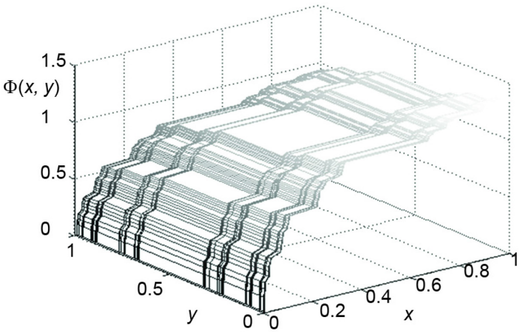

Figure 1. The exact solution of the 3-D diffusion model in fractal heat transfer when α = ln2/ln3, and τ = 0

Conclusions

In this work, the LFADM has been successfully employed to solve the 3-D diffusion model in fractal heat transfer involving LFD. The obtained solution is a non-differentiable function, which is defined on Cantor function and it discontinuously depend on the LFD.

Nomenclature

x, y, z – space co-ordinates, [m]
$\Phi(x, y, z, \tau)$ –the concentration distribution, [–]

Greek symbols

α – time fractal dimensional order, [–]
τ – time, [s]

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Appendix A

The local fractional derivative (local fractional differential operator) of $\Psi(x)$ of order α at $x = x_0$ is defined as [1, 2, 10-12]:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\Psi(x)\Big|_{x=x_0} = \Psi^{(\alpha)}(x) = \lim_{x\to x_0}\frac{\Delta^\alpha[\Psi(x)-\Psi(x_0)])}{(x-x_0)^\alpha} \tag{A1}$$

where $\Delta^\alpha[\Psi(x)-\Psi(x_0)] \cong \Gamma(\alpha+1)[\Psi(x)-\Psi(x_0)]$.

Its inverse operator (local fractional integral operator) $\psi(x)$ of order α in the interval $[\xi,\zeta]$ is given as [1, 2, 10-12]:

$$\xi I_\zeta^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)}\int_\xi^\zeta \psi(\tau)(\mathrm{d}\tau)^\alpha = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta\tau\to 0}\sum_{j=0}^{N-1} f(\tau_j)(\Delta\tau_j)^\alpha \tag{A2}$$

where the partitions of the interval $[\xi,\zeta]$ are denoted as (τ_j,τ_{j+1}), with $\Delta\tau_j = \tau_{j+1}-\tau_j$, $\tau_0 = a$, $\tau_N = b$, and $\Delta\tau = \max\{\Delta\tau_0,\Delta\tau_1,.....\}$, $j = 0,...,N-1$.

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