A NEW NUMERICAL TECHNIQUE FOR SOLVING FRACTIONAL SUB-DIFFUSION AND REACTION SUB-DIFFUSION EQUATIONS WITH A NON-LINEAR SOURCE TERM

by

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In this paper, we are concerned with the fractional sub-diffusion equation with a non-linear source term. The Legendre spectral collocation method is introduced together with the operational matrix of fractional derivatives (described in the Caputo sense) to solve the fractional sub-diffusion equation with a non-linear source term. The main characteristic behind this approach is that it reduces such problems to those of solving a system of algebraic equations which greatly simplifying the problem. In addition, the Legendre spectral collocation methods applied also for a solution of the fractional reaction sub-diffusion equation with a non-linear source term. For confirming the validity and accuracy of the numerical scheme proposed, two numerical examples with their approximate solutions are presented with comparisons between our numerical results and those obtained by other methods.

Key words: *local fractional variational iteration method, diffusion equation, non-differentiable solution, local fractional derivative*

Introduction

In recent decades fractional differential equations (FDE – differential equations with non-integer arbitrary order) have gained much attention for their many applications in engineering and physics [1-7]. Finding numerical methods to solve FDE has become the focus of many researchers study, for instance, spectral tau method [8], spectral collocation method [9], wavelet operational method [10, 11], Haar wavelet method [12], reproducing kernel function [13], spline collocation method [14], and other numerical methods [15-22].

Over the last four decades, spectral methods [23-32] have been developed rapidly. They have a good reputation compared with others numerical tools due to their widely applications in many fields. Besides, spectral methods have high accuracy; they also have exponential rates of convergence. Recently, the classical spectral methods have been developed to obtain accurate solutions for linear and non-linear FDE, for instance, [33]. Spectral methods with the help of operational matrices of fractional derivatives have been considered for solving FDE [33, 34]. This is not all, spectral methods have been used also for the fractional integro-differential equations [35, 36] and for the partial FDE [37, 38].

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Recently, models have been proposed to describe processes that become less anomalous as time progresses by the inclusion of a secondary fractional time derivative acting on a diffusion operator with a non-linear source term [39, 40]:

$$\frac{\partial y(x,t)}{\partial t}=\left(A\frac{\partial^{1-\gamma_1}}{\partial t^{1-\gamma_1}}+B\frac{\partial^{1-\gamma_2}}{\partial t^{1-\gamma_2}}\right)\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right]+f\left[y(x,t),x,t\right],\quad 0<x\le L,\quad 0<t\le T \tag{1}$$

with boundary conditions:

$$y(0,t)=\varphi_1(t),\qquad y(L,t)=\varphi_2(t),\qquad 0<t\le T \tag{2}$$

and initial condition:

$$y(x,0)=\varphi(x),\qquad 0<x\le L \tag{3}$$

where $0<\gamma_1,\gamma_2\le 1, A,$ and B are positive constants, and $f[y(x,t),x,t]$ is the non-linear source term. The symbols $\partial^{1-\gamma_1}/\partial t^{1-\gamma_1}$ and $\partial^{1-\gamma_2}/\partial t^{1-\gamma_2}$ are the Caputo fractional derivative operator and are defined as:

$$\frac{\partial^{1-\gamma_1}}{\partial t^{1-\gamma_1}}y(x,t)=\frac{1}{\Gamma(\gamma_1)}\int_0^t (t-\varepsilon)^{\gamma_1-1}\frac{\mathrm{d}y(x,\varepsilon)}{\mathrm{d}\varepsilon}\mathrm{d}\varepsilon$$

$$\frac{\partial^{1-\gamma_2}}{\partial t^{1-\gamma_2}}y(x,t)=\frac{1}{\Gamma(\gamma_2)}\int_0^t (t-\varepsilon)^{\gamma_2-1}\frac{\mathrm{d}y(x,\varepsilon)}{\mathrm{d}\varepsilon}\mathrm{d}\varepsilon$$

where $\Gamma(.)$ is the gamma function.

In [39], Liu *et al.* introduced a conditionally stable difference method to solve the fractional sub-diffusion equation (FSDE) with a non-linear source term (1)-(3). Recently, Mohebbi *et al.* [40] proposed a high order difference scheme for a solution of (1)-(3). Moreover, Bhrawy *et al.* [41] proposed a new tau spectral method based on Jacobi operational matrix for solving time fractional diffusion-wave equations. More recently, Bhrawy and Zaky [42] proposed a Jacobi tau approximation for solving multi-term time-space fractional partial differential equations.

On the other hand, Abbaszade and Mohebbi [43] applied the forth-order difference scheme for the numerical solution of the fractional reaction sub-diffusion equation (FRSDE) with a non-linear source term:

$$\frac{\partial y(x,t)}{\partial t}=\frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}\left[A\frac{\partial^2 y(x,t)}{\partial x^2}-By(x,t)\right]+f[y(x,t),x,t],\quad 0<x\le L,\quad 0<t\le T \tag{4}$$

with boundary conditions:

$$y(0,t)=\varphi_1(t),\qquad y(L,t)=\varphi_2(t),\qquad 0<t\le T \tag{5}$$

and initial condition:

$$y(x,0)=\varphi(x),\qquad 0<x\le L \tag{6}$$

where $0<\gamma\le 1$ and the symbol $\partial^{1-\gamma}/\partial t^{1-\gamma}$ is defined as:

$$\frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}} y(x, t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\varepsilon)^{\gamma-1} \frac{\mathrm{d}y(x, \varepsilon)}{\mathrm{d}\varepsilon} \mathrm{d}\varepsilon$$

In this work, we develop two efficient algorithms to solve spectrally the FSDE with a non-linear source term (1)-(3) and the FRSDE with a non-linear source term (4)-(6). These algorithms are based basically on a new Legendre spectral collocation method (LSCM) combined with the fractional derivatives operational matrices of shifted Legendre polynomials. For confirming the validity and accuracy of the proposed spectral schemes, we present three numerical examples combined with comparisons with other methods. Moreover, the present method is more accurate than compact finite difference scheme [40], finite difference method [44] and forth-order compact finite difference scheme [43].

This article is organized as follow. In next section, we introduce some properties of Legendre polynomials and state the operational matrix of fractional derivatives. In section *Legendre spectral collocation method*, the operational matrix of fractional derivatives is used with the help of the LSCM to introduce an approximate solution for the FSDE with a non-linear source term and the FRSDE with a non-linear source term. The section *Numerical examples* contains three numerical examples combined with comparisons between our results and those obtained by other methods are introduced. And the final section are the *Conclusions*.

Shifted Legendre polynomials

Assuming that the Legendre polynomial of degree j is denoted by $P_j(z)$, defined on the interval (–1, 1). Then $P_j(z)$ may be generated by the recurrence equation:

$$P_{j+1}(z) = \frac{2j+1}{j+1} z P_j(z) - \frac{j}{j+1} P_{j-1}(z), \qquad j = 1, 2, \cdots,$$
$$P_0(z) = 1, \qquad P_1(z) = z$$

Introducing $z = 2x - 1$, Legendre polynomials are defined on the interval $(0, 1)$ that may be called shifted Legendre polynomials $P_j^*(x)$ that generated using the recurrence equation:

$$P_{j+1}^*(x) = \frac{2j+1}{j+1}(2x-1) P_j^*(x) - \frac{j}{j+1} P_{j-1}^*(x), \qquad j = 1, 2, \cdots,$$
$$P_0^*(x) = 1, \qquad P_1^*(x) = 2x - 1$$

The orthogonality relation is:

$$\int_0^1 P_i^*(x) P_j^*(x) \mathrm{d}x = \begin{cases} \dfrac{1}{2j+1}, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases} \tag{7}$$

The explicit analytical form of shifted Legendre polynomial $P_j^*(x)$ of degree j may be written as:

$$P_j^*(x) = \sum_{k=0}^{j} (-1)^{j+k} \frac{(j+k)!\, x^k}{(j-k)!\, (k!)^2} \tag{8}$$

and this in turn, enables one to get:

$$P_j^*(0) = (-1)^j, \qquad P_j^*(1) = 1$$

Any square integrable function $y(x)$ defined on the interval $(0, 1)$, may be expressed in terms of shifted Legendre polynomials as:

$$y(x) = \sum_{j=0}^{\infty} a_j P_j^*(x)$$

from which the coefficients a_j are given by:

$$a_j = (2j+1)\int_0^1 y(x)P_j^*(x)\mathrm{d}x, \qquad j = 0, 1, \cdots \tag{9}$$

If we approximate $y(x)$ by the first $(n + 1)$-terms, then we can write:

$$y_n(x) = \sum_{j=0}^{n} a_j P_j^*(x) \tag{10}$$

which alternatively may be written in the matrix form:

$$y_n(x); \quad A^T \Delta_n(x) \tag{11}$$

with

$$A^T \equiv [a_0, a_1, \cdots, a_n], \quad \Delta_n(x) \equiv [P_0^*(x), P_1^*(x), \cdots, P_n^*(x)]^T \tag{12}$$

Similarly, let $y(x, t)$ be an infinitely differentiable function defined on $0 < x \leq 1$ and $0 < t \leq 1$. Then it is possible to express as:

$$y_{m,n}(x, t); \sum_{i=0}^{m}\sum_{j=0}^{n} y_{ij} P_i^*(t) P_j^*(x) = \Delta_m^T(t) Y \Delta_n(x) \tag{13}$$

with

$$Y = \begin{pmatrix} y_{00} & y_{01} & \cdots & y_{0n} \\ y_{10} & y_{11} & \cdots & y_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ y_{m0} & y_{m1} & \cdots & y_{mn} \end{pmatrix}$$

and

$$y_{ij} = (2i+1)(2j+1)\int_0^1\int_0^1 y(x, t) P_i^*(t)\, P_j^*(x)\mathrm{d}x\mathrm{d}t, \quad i = 0, 1, \cdots, m, \qquad j = 0, 1, \cdots, n \tag{14}$$

Theorem. The Caputo fractional derivative of order γ of the shifted Legendre polynomial vector $\Delta_m(t)$ is given by [34]:

$$D^\gamma \Delta_m(t) = D_\gamma \Delta_m(t) \tag{15}$$

where D_γ is the $(m+1)\times(m+1)$ operational matrix of fractional derivative of order γ defined by:

$$D_\gamma = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\gamma\rceil}^{\lceil\gamma\rceil}\theta(\lceil\gamma\rceil,0,k) & \sum_{k=\lceil\gamma\rceil}^{\lceil\gamma\rceil}\theta(\lceil\gamma\rceil,1,k) & \cdots & \sum_{k=\lceil\gamma\rceil}^{\lceil\gamma\rceil}\theta(\lceil\gamma\rceil,m,k) \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\gamma\rceil}^{i}\theta(i,0,k) & \sum_{k=\lceil\gamma\rceil}^{i}\theta(i,1,k) & \cdots & \sum_{k=\lceil\gamma\rceil}^{i}\theta(i,m,k) \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\gamma\rceil}^{m}\theta(m,0,k) & \sum_{k=\lceil\gamma\rceil}^{m}\theta(m,1,k) & \cdots & \sum_{k=\lceil\gamma\rceil}^{m}\theta(m,m,k) \end{bmatrix}$$

where

$$\theta(i,j,k) = (2j+1)\sum_{l=0}^{j}\frac{(-1)^{i+j+k+l}(i+k)!(l+j)!}{(i-k)!k!\Gamma(k-\gamma+1)(j-l)!(l!)^2(k+l-\gamma+1)}$$

Legendre spectral collocation method

In this section, the Legendre operational matrix of fractional derivatives is applied with the help of Legendre collocation-spectral method to solve the FSDE with a non-linear source term (1)-(3) and the FRSDE with a non-linear source term (4)-(6).

Fractional sub-diffusion equation with a non-linear source term

From eq. (1) and making use of the initial condition (3), easily we can rewrite problem (1)-(3) as in the form:

$$\frac{\partial y(x,t)}{\partial t} = \left(A\frac{\partial^{1-\gamma_1}}{\partial t^{1-\gamma_1}} + B\frac{\partial^{1-\gamma_2}}{\partial t^{1-\gamma_2}}\right)\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right] + y(x,0) - \varphi(x) + f[y(x,t),x,t] \tag{16}$$

with boundary conditions (2).

Now, we approximate $y(x,t)$ by the shifted Legendre polynomials as:

$$y_{m,n}(x,t) = \Delta_m^T(t)Y\Delta_n(x) \tag{17}$$

where Y is an unknown coefficients $(m+1)\times(n+1)$ matrix. Using eqs. (15) and (17), we can write:

$$\frac{\partial y(x,t)}{\partial t} = \Delta_m^T(t)D_1^T Y\Delta_n(x)$$

$$y(x,0) = \Delta_m^T(0)Y\Delta_n(x)$$

$$\frac{\partial^{1-\gamma_1}}{\partial t^{1-\gamma_1}}\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right] = \Delta_m^T(t) D_{1-\gamma_1}^T Y D_2 \Delta_n(x)$$

$$\frac{\partial^{1-\gamma_2}}{\partial t^{1-\gamma_2}}\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right] = \Delta_m^T(t) D_{1-\gamma_2}^T Y D_2 \Delta_n(x) \tag{18}$$

By substituting (17) and (18) in eq. (16), we get:

$$\Delta_m^T(t)(D_1^T Y - A D_{1-\gamma_1}^T Y D_2 - B D_{1-\gamma_2}^T Y D_2)\Delta_n(x) =$$

$$\Delta_m^T(0) Y \Delta_n(x) - \varphi(x) + f[\Delta_m^T(t) Y \Delta_n(x), x, t] \tag{19}$$

We collocate eq. (19) at $(m+1)\times(n-1)$ points, as:

$$\Delta_m^T(t_i)(D_1^T Y - A D_{1-\gamma_1}^T Y D_2 - B D_{1-\gamma_2}^T Y D_2)\Delta_n(x_j) =$$

$$\Delta_m^T(0) Y \Delta_n(x_j) - \varphi(x_j) + f[\Delta_m^T(t_i) Y \Delta_n(x_j), x_j, t_i] \tag{20}$$

where $t_i, i = 0,1,\cdots,m$ are the roots of $P_{m+1}^*(t)$, while $x_j, j = 0,1,\cdots,n-2$ are the roots of $P_{n-1}^*(x)$, that generates a system of $(m+1)\times(n-1)$ non-linear algebraic equations in the unknown expansion coefficients, $y_{ij}, i = 0,1,\cdots,m; j = 0,1,\cdots,n-2$ and the rest of this system is obtained from the boundary condition (2), as:

$$\Delta_m^T(t) Y \Delta_n(0) = \varphi_1(t_i), \quad \Delta_m^T(t) Y \Delta_n(1) = \varphi_2(t_i), \quad i = 0,1,\cdots,m \tag{21}$$

This generate $(m+1)\times(n+1)$ non-linear algebraic equations which can be solved using Newton's iterative method. Consequently $y_{m,n}(x,t)$ given in (13) can be calculated.

Fractional reaction sub-diffusion equation with a non-linear source term

Similarly, as in the previous section, (4) may be written in the form:

$$\frac{\partial y(x,t)}{\partial t} = \frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}\left[A\frac{\partial^2 y(x,t)}{\partial x^2} - B y(x,t)\right] + y(x,0) - \varphi(x) + f[y(x,t),\ x,\ t] \tag{22}$$

with boundary conditions (5).

After approximating $y(x,t)$ by the shifted Legendre polynomials, as in eq. (17) and making use (15), we can write:

$$\frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right] = \Delta_m^T(t) D_{1-\gamma}^T Y D_2 \Delta_n(x)$$

$$\frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}[y(x,t)] = \Delta_m^T(t) D_{1-\gamma}^T Y \Delta_n(x) \tag{23}$$

By substituting eqs. (17), (18), and (23) in eq. (22) we get:

$$\Delta_m^T(t)(D_1^T Y - AD_{1-\gamma}^T YD_2 + BD_{1-\gamma}^T Y)\Delta_n(x) =$$

$$= \Delta_m^T(0)Y\Delta_n(x) - \varphi(x) + f[\Delta_m^T(t)Y\Delta_n(x), x, t] \tag{24}$$

Similarly, as in the previous section, a system of $(m+1)\times(n-1)$ non-linear algebraic equations in the unknown expansion coefficients, $y_{ij}, i = 0,1,\cdots,m; j = 0,1,\cdots,n-2$ is generated by collocating eq. (24) at $(m+1)\times(n-1)$ points, as:

$$\Delta_m^T(t_i)(D_1^T Y - AD_{1-\gamma}^T YD_2 + BD_{1-\gamma}^T Y)\Delta_n(x_j) =$$

$$\Delta_m^T(0)Y\Delta_n(x_j) - \varphi(x_j) + f[\Delta_m^T(t_i)Y\Delta_n(x_j), x_j, t_i] \tag{25}$$

and the rest of this system is obtained from the boundary condition (5), as in (21). This generates $(m+1)\times(n+1)$ non-linear algebraic equations which can be solved using Newton's iterative method. Consequently $y_{m,n}(x,t)$ given in eq. (10) can be calculated.

Numerical examples

In order to clarify the validity and effectiveness of the presented algorithms, the numerical results of three numerical examples are introduced in this section, also comparisons with other numerical methods are made.

Example 1

We consider the FSDE with a non-linear source term studied in [40]:

$$\begin{gathered}
\frac{\partial y(x,t)}{\partial t} = \frac{1}{2}\left(\frac{\partial^{1-\gamma_1}}{\partial t^{1-\gamma_1}} + \frac{\partial^{1-\gamma_2}}{\partial t^{1-\gamma_2}}\right)\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right] + f[y(x,t), x, t] \\
y(x,0) = 0, \quad 0 < x \le 1 \\
y(0,t) = t^2, \quad y(1,t) = \mathrm{e}t^2, \qquad 0 < t \le 1
\end{gathered} \tag{26}$$

where the non-linear source term:

$$f[y(x,t), x, t] = y^3(x,t) + \mathrm{e}^x\left[2t - t^6\mathrm{e}^{2x} - \frac{t^{1+\gamma_1}}{\Gamma(2+\gamma_1)} - \frac{t^{1+\gamma_2}}{\Gamma(2+\gamma_2)}\right]$$

and exact solution $y(x,t) = t^2\mathrm{e}^x$.

In [40], the high-order compact finite difference scheme (H-OCFDS) is implemented to solve this problem with various choices of h and τ, where h and τ are space and time step sizes, respectively.

In order to show that the presented method is more accurate than the H-OCFDS, comparisons between the LSCM results and those obtained in [40] are introduced for different values of γ_1 and γ_2. In tab. 1, we introduce the maximum absolute errors for various choices of $m, (m = n)$ and those achieved using the H-OCFDS [40] for different values of γ_1 and γ_2. From tab. 1, it is clear that the LSCM is more accurate than the H-OCFDS [40].

Table 1. Comparing of LSCM with H-OCFDS [40] at various choices of γ_1 and γ_2 for Example 1

γ_1	γ_2	$m = 6$	$m = 8$	$m = 10$	$m = 12$	$h = 1/32$	$h = 1/32$
		$n = 6$	$n = 8$	$n = 10$	$n = 12$	$\tau = 1/640$	$\tau = 1/204/8$
		LSCM				H-OCFDS [40]	
0.45	0.75	$5.95\cdot10^{-6}$	$5.88\cdot10^{-9}$	$1.41\cdot10^{-11}$	$5.32\cdot10^{-14}$	$1.20\cdot10^{-4}$	
0.35	0.65	$5.56\cdot10^{-6}$	$4.89\cdot10^{-9}$	$1.17\cdot10^{-11}$	$5.19\cdot10^{-14}$	$9.54\cdot10^{-5}$	
0.95	0.15	$6.25\cdot10^{-4}$	$5.32\cdot10^{-9}$	$1.03\cdot10^{-11}$	$1.14\cdot10^{-13}$		$2.00\cdot10^{-5}$
0.25	0.75	$8.56\cdot10^{-6}$	$4.79\cdot10^{-9}$	$1.09\cdot10^{-11}$	$1.20\cdot10^{-13}$		$2.61\cdot10^{-5}$

Example 2

We consider the following FESDE studied in [44]:

$$\frac{\partial y(x,t)}{\partial t} = \frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}\left[\frac{\sin(\pi\gamma)}{\pi^{12}}\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{e^{\pi}}y(x,t)\right] + f(x,t)$$

$$y(x,0) = 0, \quad 0 < x \le 1 \tag{27}$$

$$y(0,t) = 0, \quad y(1,t) = 0, \quad 0 < t \le 1$$

where the non-linear source term:

$$f(x,t) = \sin(\pi x)\left\{2t + \frac{2t^{1+\gamma}}{\Gamma(2+\gamma)}\left[\frac{\sin(\pi\gamma)}{\pi^{10}} + \frac{1}{e^{\pi}}\right]\right\}$$

and exact solution $y(x,t) = t^2\sin(\pi x)$.

In [44], the authors used the mixed parameters spline function to get two classes of finite difference methods (FDM) to solve the FRSDE (27) at different values of h, τ, and γ. In tab. 2, we compare results obtained in [44] with those achieved by the LSCM for different choices of γ. From tab. 2, it is clear that the LSCM is more accurate than the FDM [44].

Table 2. Comparing of LSCM with the second class of FDMs [44] at γ = 0.6 and 0.8 for Example 2

LSCM			2-FDM [44]			
N = M	$\gamma = 0.6$	$\gamma = 0.8$	h	τ	$\gamma = 0.6$	$\gamma = 0.8$
4	$8.7883\cdot10^{-4}$	$8.7884\cdot10^{-4}$	1/4	1/4	$2.4509\cdot10^{-1}$	$2.4438\cdot10^{-1}$
6	$9.5918\cdot10^{-6}$	$9.5904\cdot10^{-6}$	1/8	1/16	$6.1363\cdot10^{-2}$	$6.0727\cdot10^{-2}$
8	$6.6860\cdot10^{-8}$	$6.6838\cdot10^{-8}$	1/16	1/64	$1.5346\cdot10^{-2}$	$1.5158\cdot10^{-2}$
10	$3.0455\cdot10^{-10}$	$3.0426\cdot10^{-10}$	1/2	1/2	$4.8925\cdot10^{-1}$	$4.9259\cdot10^{-1}$
12	$1.0378\cdot10^{-12}$	$1.3506\cdot10^{-12}$	1/4	1/32	$3.0689\cdot10^{-2}$	$3.0322\cdot10^{-2}$
14	$3.1663\cdot10^{-13}$	$1.3962\cdot10^{-12}$	1/8	1/512	$1.4021\cdot10^{-3}$	$2.2131\cdot10^{-3}$

Conclusions

In this work, a fast and effective numerical scheme was constructed for highly accurate numerical solution for the FSDE with a non-linear source term. In the proposed approach, in order to approximate the problem and to reduce it to a solution of non-linear algebraic equations, the shifted Legendre polynomials were used as a basis function of the spectral collocation method with the help of the operational matrices of fractional derivative and integral. This approach was used also for a solution of the FRSDE with a non-linear source term. The main advantage of the proposed algorithm is, adding few terms of the shifted Legendre polynomials, a good approximation of the exact solution of the problem is achieved. Comparisons between approximate solutions using the LSCM with the exact solutions of these problems and with those achieved by other methods were introduced to confirm the validity and accuracy of our scheme.

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