A DECOMPOSITION METHOD FOR SOLVING DIFFUSION EQUATIONS VIA LOCAL FRACTIONAL TIME DERIVATIVE

by

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In this paper a decomposition method based on Daftardar-Jafari method is applied for solving diffusion equations involving local fractional time derivatives. The convergence of this method for solving these type of equations is proved.

Key words: *iterative method, local fractional derivative, diffusion equation, Daftardar-Jafari method*

Introduction

Fractional calculus and fractional dynamics have become hot topics of research in which there is rapid development and implementation in various fields of engineering and science [1-5]. Analysis of the diffusion equation in mathematical physics have been of considerable interest in the literature. Fractional diffusion equation has important applications to mathematical physics. Nigmatullin [6] has employed the fractional diffusion equation to describe diffusion in media with fractal geometry. Also this equation is used to model anomalous diffusion in plasma transport. In the recent years several authors, Jafari *et al.* [7], Chuna [8], and Saha Ray [9, 10], have solved the fractional diffusion-wave equations using different methods.

Recently, local fractional calculus theory has been used to model some nondifferentiable problems for mathematica physics [11-17] and the references therein. Our main aim in this paper is to apply a new iterative method to solve diffusion equations involving local fractional time derivatives. This method is introduced by Daftardar and Jafari [18], later referred to as the Daftardar-Jafari method (DJM) [2]. The core of this approach is to solve non-linear equations without using Adomian polynomials. In [2, 19] the convergence of this method is discussed.

Basic definitions

Definition 1. A real function $f(x)$ is said local fractional continuous on the interval (a,b) if $\forall\, x \in (a,b)\ \lim_{x\to x_0} f(x) = f(x_0)$ which is denoted by $f(x) \in C_\alpha(a,b)$.

Definition 2. Let $f(x) \in C_\alpha(a,b)$, the local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined as:

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$$D_x^\alpha f(x_0) = \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} f(x)|_{x=x_0} = f^{(\alpha)}(x) = \lim_{x \to x_0} \frac{\Delta^\alpha [f(x) - f(x_0)]}{(x - x_0)^\alpha} \tag{1}$$

where $\Delta^\alpha [f(x) - f(x_0)] \cong \Gamma(\alpha + 1)[f(x) - f(x_0)]$.

Note that the local fractional derivative of high order and the local fractional partial derivative of high order are written in the form:

$$D_x^{k\alpha} f(x) = f^{(k\alpha)}(x) = \overbrace{D_x^\alpha \cdots D_x^\alpha}^{k \text{ times}} f(x) \tag{2}$$

$$\frac{\partial^{k\alpha}}{\partial x^{k\alpha}} f(x, y) = \overbrace{\frac{\partial^\alpha}{\partial x^\alpha} \cdots \frac{\partial^\alpha}{\partial x^\alpha}}^{k \text{ times}} f(x, y) \tag{3}$$

respectively.

Definition 3. The local fractional integral operator is defined as:

$${}_aI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t)(\mathrm{d}t)^\alpha = \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha \tag{4}$$

where $\Delta t_j = t_{j+1} - t_j, \Delta t = \max\{\Delta t_0, \Delta t_1, \cdots, \Delta t_j, \cdots\}$, $[t_j, t_{j+1}]$, $j = 0, \cdots, N - 1$, $t_0 = a$, and $t_N = b$, is a partition of the interval $[a, b]$.

The following properties hold:

$$D_x^\alpha [f(x)g(x)] = [D_x^\alpha f(x)]g(x) + f(x)[D_x^\alpha g(x)]$$

$${}_aI_x^\alpha f(x)g^{(\alpha)}(x) = [f(x)g(x)]|_a^x - {}_aI_x^\alpha f^{(\alpha)}(x)g(x)$$

$$D_x^\alpha \frac{x^{k\alpha}}{\Gamma(1 + k\alpha)} = \frac{x^{(k-1)\alpha}}{\Gamma[1 + (k - 1)\alpha]}$$

$${}_0I_x^\alpha \frac{x^{k\alpha}}{\Gamma(1 + k\alpha)} = \frac{x^{(k+1)\alpha}}{\Gamma[1 + (k + 1)\alpha]} \tag{5}$$

Definition 4. In fractal space, the Mittage-Leffler function, sine and cosine functions are defined on Cantor sets are defined as:

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{\alpha n}}{\Gamma(1 + \alpha n)}, \quad 0 < \alpha \le 1 \tag{6}$$

$$\sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)\alpha}}{\Gamma[1 + (2n + 1)\alpha]}, \quad 0 < \alpha \le 1 \tag{7}$$

$$\cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n\alpha}}{\Gamma[1 + 2n\alpha]}, \quad 0 < \alpha \le 1 \tag{8}$$

Note that:

$$\frac{\mathrm{d}^{\alpha}E_{\alpha}(x^{\alpha})}{\mathrm{d}x^{\alpha}}=E_{\alpha}(x^{\alpha}) \tag{9}$$

$$\frac{\mathrm{d}^{\alpha}E_{\alpha}(kx^{\alpha})}{\mathrm{d}x^{\alpha}}=kE_{\alpha}(kx^{\alpha}) \tag{10}$$

Analysis of the method

In this section, a decomposition method based on Daftardar-Jafari method is applied for solving diffusion and heat equation involving local fractional derivative [18]. This decomposition of the non-linear function is quite different from that of Adomain decomposition. we investigate the general form of non-linear local fractional partial equation:

$$D_t^{\alpha}u(\overline{x},t)=\sum_{i=1}^{n}\psi_i(\overline{x},t)\frac{\partial^2 u(\overline{x},t)}{\partial x_i^2}+\varphi(\overline{x},t)u^m(\overline{x},t) \tag{11}$$

$$u(\overline{x},0)=g(\overline{x}) \tag{12}$$

where $m=0,1,2,\ldots$, $\overline{x}=(x_1,\ldots x_n)$, and $\psi_i(\overline{x},t)\in C_{\alpha}$. For $m=0$ or 1, and $0<\alpha<1$, eq. (11) represents linear fractional diffusion equation (homogeneous if $\varphi(\overline{x},t)=0$ and non-homogeneous otherwise). It represents non-linear diffusion equation for $m = 2, 3,\ldots$

When $L_t^{(\alpha)}=D_t^{\alpha}$ and by applying $L_t^{(-\alpha)}$ to eq. (11), we get:

$$u(\overline{x},\cdots t)=u(\overline{x},0)+L_t^{(-\alpha)}[\varphi(\overline{x},t)u^m(\overline{x},t)]+L_t^{(-\alpha)}\left[\sum_{i=1}^{n}\psi_i(\overline{x},t)\frac{\partial^2 u(\overline{x},t)}{\partial x_i^2}\right] \tag{13}$$

Following Daftardar-Jafari method, the unknown function can be shown in terms of an infinite series as:

$$u=\sum_{i=0}^{\infty}u_i \tag{14}$$

For $m=0$ we set:

$$f(\overline{x})=u(\overline{x},0)+L_t^{(-\alpha)}[\varphi(\overline{x},t)]$$
$$N(u)=L_t^{(-\alpha)}\sum_{i=1}^{n}\psi_i(\overline{x},t)\frac{\partial^2 u(\overline{x},t)}{\partial x_i^2} \tag{15}$$

and for other values of $m = 1, 2, \ldots$ we set:

$$f(\overline{x})=u(\overline{x},0)$$
$$N(u)=L_t^{(-\alpha)}[\varphi(\overline{x},t)u^m(\overline{x},t)]+L_t^{(-\alpha)}\sum_{i=1}^{n}\psi_i(\overline{x},t)\frac{\partial^2 u(\overline{x},t)}{\partial x_i^2} \tag{16}$$

The non-linear function N can be decomposed as:

$$N\left(\sum_{i=0}^{\infty}u_i\right)=N(u_0)+\sum_{i=0}^{\infty}\left(N\sum_{j=0}^{i}u_j-N\sum_{j=0}^{i-1}u_j\right) \tag{17}$$

Substituting eqs. (14) and (17) in eq. (13), we have:

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=0}^{\infty}\left(N\sum_{j=0}^{i} u_j - N\sum_{j=0}^{i-1} u_j \right)$$

then we have the recurrence relations:

$$\begin{cases} u_0 = f \\ u_1 = N(u_0) \\ u_{n+1} = N(u_0 + \ldots + u_n) - N(u_0 + \ldots + u_{n-1}), \quad n = 1, 2, \cdots. \end{cases} \tag{18}$$

and

$$u = f + \sum_{i=1}^{\infty} u_i \tag{19}$$

are calculated.

*Theorem 1***.** If the series solution defined in eq. (19) is convergent, then it converges to an exact solution of eq. (11).

Proof. The truncated series $\sum_{i=0}^{m} u_i$ is used as an approximation to the solution $u(t)$ of eq. (11), using the above we have:

$$\sum_{i=0}^{m} u_i = f + N\sum_{i=0}^{m} u_i \tag{20}$$

Taking limits of eq. (20), it gains:

$$u = \lim_{m\to\infty}\sum_{i=0}^{m} u_i = \lim_{m\to\infty}\left(f + N\sum_{i=0}^{m} u_i \right) = \lim_{m\to\infty} f + \lim_{m\to\infty} N\sum_{i=0}^{m} u_i = f + N\lim_{m\to\infty}\sum_{i=0}^{m} u_i = f + N(u)$$

Hence u is the solutions of eq. (11) and the proof is complete.

Illustrative examples

To demonstrate the effectiveness of the method we consider here some fractional diffusion and heat equations with local fractional time derivative.

Example 1. Consider the diffusion equation with local fractional time derivative:

$$D_t^{\alpha} u(x,t) = -\frac{\partial^2 u(x,t)}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \tag{21}$$

$$u(x,0) = \mathrm{e}^{-x}, \quad \alpha \in (0,1)$$

By applying $L_t^{(-\alpha)} = I_t^{\alpha}$ on both side of eq. (21) and using initial condition, we get:

$$u(x,t) = \mathrm{e}^{-x} - I_t^{\alpha} \frac{\partial^2 u(x,t)}{\partial x^2} \tag{22}$$

Following this method, $u(x,t)$ is represented as:

$$u=\sum_{i=0}^{\infty}u_i \tag{23}$$

and we get:

$$N(u)=-I_t^{\alpha}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{24}$$

where the non-linear function N can be decomposed as:

$$N\sum_{i=0}^{\infty}u_i=N(u_0)+\sum_{i=0}^{\infty}\left(N\sum_{j=0}^{i}u_j-N\sum_{j=0}^{i-1}u_j\right) \tag{25}$$

According to the Daftardar-Jafari method and applying, we obtain:

$$\begin{cases} u_0(x,t)=\mathrm{e}^{-x} \\ u_1(x,t)=N(u_0) \\ u_{n+1}(x,t)=N\sum_{j=0}^{n}u_j-N\sum_{j=0}^{n-1}u_j \quad n=1,2,... \end{cases}$$

In the first iteration we have:

$$u_1(x,t)=-I^{\alpha}\left(\frac{\partial^2 u_0}{\partial x^2}\right)=-\frac{1}{\Gamma(\alpha+1)}\int_0^t \mathrm{e}^{-x}(\mathrm{d}t)^{\alpha}=-\mathrm{e}^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)} \tag{26}$$

$$u_2(x,t)=\mathrm{e}^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{27}$$

In similar manner, we can derive the other approximation. Finally, the compact solution becomes:

$$u(x,t)=\mathrm{e}^{-x}\left[1-\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\ldots\right]=\mathrm{e}^{-x}\left[1+\sum_{n=1}^{\infty}\frac{(-t^{\alpha})^n}{\Gamma(n\alpha+1)}\right]=\mathrm{e}^{-x}E_{\alpha}(-t^{\alpha}) \tag{28}$$

where $E_{\alpha}(kz^{\alpha})=\sum_{n=0}^{\infty}(k^n z^{\alpha n})/\Gamma(\alpha n+1)$ denotes Mittag-Leffler function defined on Cantor sets.

Example 2. Consider the fractional diffusion equation with local fractional time derivative:

$$D_t^{\alpha}u(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2}+u(x,t) \tag{29}$$

$$u(x,0)=\cos(\pi x), \quad \alpha\in(0,1)$$

According similar procedure as in the previous example, we can structure the same local fractional iteration procedure of eq. (18):

$$\begin{cases} u_0(x,t) = u(x,0) \\ u_1(x,t) = N(u_0) \\ u_{n+1}(x,t) = N\sum_{j=0}^{n} u_j - N\sum_{j=0}^{n-1} u_j \quad n = 1, 2, \ldots \end{cases} \tag{30}$$

Making use of eq. (30), we present:

$$u_0(x,t) = \cos(\pi x) \tag{31}$$

$$u_1(x,t) = [1-\pi^2]\cos(\pi x)\frac{t^\alpha}{\Gamma(\alpha+1)} \tag{32}$$

$$u_2(x,t) = [1-\pi^2]^2\cos(\pi x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{33}$$

Proceeding in this manner, we can derive the other approximation. Thus, the final solution reads:

$$u(x,t) = \cos(\pi x) + [1-\pi^2]\cos(\pi x)\frac{t^\alpha}{\Gamma(\alpha+1)} + [1-\pi^2]^2\cos(\pi x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots =$$

$$= \cos(\pi x)\left(1 + \frac{[1-\pi^2]t^\alpha}{\Gamma(\alpha+1)} + \frac{[1-\pi^2]^2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots\right) = \cos(\pi x)E_\alpha([1-\pi^2]t^\alpha) \tag{34}$$

where $E_\alpha(kz^\alpha)$ denotes the Mittag-Leffler function defined on Cantor sets.

Conclusion

In this work, we applied the Daftardar-Jafari method to solve the diffusion and heat equations involving local fractional derivative. The results show that this method is accurate and effective and can be used for non-linear local fractional differential equations. This method have an advantage over the Adomian decomposition method in that the Daftardar-Jafari method can solve non-linear problems without using Adomian polynomials.

Mathematica has been used for computations and programming in this paper.

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