EXTENSION OF FRAGMENTATION PROCESS IN A KINETIC-DIFFUSIVE-WAVE SYSTEM

by

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Alternative methods are used to set conditions and investigate, in the space $L_1(R^3 \times R_+ \; mdmdx)$ the well-posedness of a fractional fragmentation process in a kinetic-diffusive-wave medium. In the analysis, three separate models of diffusion are studied. Techniques like separation of variables and subordination principle are used to finally prove that the Cauchy problem for fractional fragmentation dynamics in a kinetic-diffusive-wave system is well-posed and admits a solution operator that is positive and contractive. This work brings a contribution that may lead to the full explanation of strange phenomena like shattering and sudden appearance of an infinite number of particles in a system that occur in the dynamics of fragmentation process and which remain partially unsolved.

Key words: *fragmentation, kinetic-diffusive-wave process, fractional Cauchy problem, well-posedness, solution operators*

Introduction and model's description

Evolution equations with derivative of fractional order and their applications in real life have been intensively analyzed in a large number of works [1-8]. For instance, sub- and super-diffusion processes well applicable in reaction-diffusion systems and which is a typical application of fractional derivatives have been analyzed in [3] where the authors used the concept of fractional Laplacian operator in the theory of Levy flights. The author in [4] successfully applied fractional derivatives to provide a bio-mathematical analysis of the model of fragmentation dynamics with derivative of fractional order describing the cellulose degradation. In the same manner, other authors [9-12] successfully generalized, in various ways, traditional derivatives of integer order to derivatives of fractional order. The model we propose to investigate here is a generalized kinetic-type evolution equation describing the fragmentation of clusters in the diffusive wave process.

Recall that fragmentation equation with first order time derivative has been comprehensively investigated in many research works [13-17]. Conservative and non-conservative regimes for fragmentation equations have been thoroughly investigated in [13, 18], and, in particular, the breach of the mass conservation law, also known as shattering phenomena has been attributed to a phase transition creating a dust of zero-size particles with nonzero mass. But it is beyond the resolution of models usually analyzed. This is one of the reasons why

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there are increasing new approaches in order to extend classical models to those with fractional derivative, investigate them with different methods and provide results with a broader outlook. This extension can be done in many ways. For example, the authors in [19] successfully generalized the advection-dispersion equation (to the fractional one) by using various techniques including the well-known action of Fourier transform on integer derivatives to rational order. With this in mind, and following the same approach as [10, 12, 20], our generalized model can be obtained by evaluating in one side, the changing due to fragmentation and diffusion and the other side, the fractional variation that replicates the real picture of that changing. Hence, we obtain the next Cauchy problem:

$$D_t^\alpha[u(x,m,t)] = D_x[\mathrm{d}(x,m)D_x u(x,m,t)] - a(x,m)u(x,m,t) + \\ + \int_m^\infty a(x,y)b(m,y)u(x,y,t)\,\mathrm{d}y \tag{1}$$

subject to the initial conditions:

$$u(x,m,0) = f(x,m) \\ D_N u\,|_\Omega = 0 \tag{2}$$

with $0<\alpha<1,\ t,\ m>0,\ x\in\Omega\subset R^3$. Here:

$$ {}_0^C D_t^\alpha[u(x,m,t)] = \frac{\partial^\alpha}{\partial t^\alpha}u(x,m,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-r)^{-\alpha}\frac{\partial}{\partial r}u(x,r)\mathrm{d}r \tag{3}$$

with $0 \le \alpha < 1$ is the fractional derivative of $u(x, m, t)$ in the sense of Caputo [21], with Γ the gamma function. For reasons of simplicity we note ${}_0^C D_t^\alpha = D_t^\alpha$. The differential operator D_x represents $\partial/\partial x$ and $\mathrm{d}(x, m)$ is the diffusion coefficient. D_N is the first order derivative in the direction of the outward normal N to the boundary $\partial\Omega$ where Ω is an open subset of R^3 with a C^2 boundary $\partial\Omega$. This model describes the dynamics of clusters with mass size m and position x using the mass-position density function $u(t, x, m)$, where $u : R_+ \times \Omega \times R_+ \to R_+$. The quantity $a(x, m)$ represents the fragmentation rate which describes the ability of clusters of size m and position x to split into smaller particles. The non-negative measurable function $b(s, m)$ defined on $R \times R$ is the distribution kernel providing the system with the expected number of daughter particles of size m after the breakup of a mass s-cluster located at position x. The function $f(x, m)$ represents the density of m-clusters at position x at the beginning ($t = 0$). It is assumed that the space variable x can take any value in whole Ω.

There is need to recall necessary and useful assumptions that will be relevant for our analysis in the framework of fragmentation process. It is assumed that the conservation law is verified:

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = 0 \tag{4}$$

where $N(t) = \int_{R^3}\int_0^\infty u(t,x,m)m\mathrm{d}m\mathrm{d}x$, stands for the total mass of the ensemble or the total number of individuals in the space. This law is motivated by the fact that the total number of individuals in a population is not modified by interactions among clusters. Hence, the mass is expected to be a conserved quantity and the appropriate space for it is:

$$H := L_1(R^3 \times R_+, m\mathrm{d}m\mathrm{d}x),$$

endowed with the norm ||.||. We assume that $f \in H$ and for each $t \geq 0$, the function $(x, m) \to \to u(x, m) = u(t, x, m)$ is from the space H.

Furthermore, it is obvious that cluster of size $m \leq s$ cannot breakup to give a s-cluster. So, the function $b(x, s, m)$ has its support in the set:

$$R^3 \times \{(s,m) \in R_+ \times R_+ : m < s\} \tag{5}$$

We also have, for any $s > 0, x \in R^3$:

$$\int_0^s mb(s,m)\mathrm{d}m = s \tag{6}$$

meaning that the sum of masses of all daughter particles, issued from the breakup of a mass s-particle, should again give s. If in H, we define from the expressions on the right-hand side of (1), the operators B and E by:

$$[Ev](x,m) := a(x,m)v(x,m), \quad D(E) := \{v \in H : av \in H\} \tag{7}$$

$$[Bv](x,m) := \int_m^\infty a(x,y)b(m,y)v(x,y)\mathrm{d}y, \quad D(B) := D(E) \tag{8}$$

Lemma 1. $[E + B, D(E)]$ is a well-defined operator.
Proof. [15, Lemma 1]

Solvability of some fractional diffusion models

To investigate the solvability of some evolution equations like the one involved in the diffusion process, we need to recall some additional useful concepts. Firstly, let us mention some properties of the Mittag-Leffler function happening in complex relaxation processes. Consider the Mittag-Leffler function [22, 23]:

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha + 1)} \tag{9}$$

Using the Laplace transform L and its inverse, it can be proved, see [5, 24], that:

$$\begin{aligned}
E_\alpha(-x) &= \frac{2x}{\pi}\int_0^\infty \frac{E_{2\alpha}(-r^2)}{x^2 + r^2}\mathrm{d}r, \qquad 0 < \alpha < 1 \\
&= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty \frac{r^{\alpha-1}\mathrm{e}^{-x^{\frac{1}{\alpha}}r}\mathrm{d}r}{r^{2\alpha} + 2r^\alpha \cos(\alpha\pi) + 1}, \qquad 0 < \alpha < 1 \\
&= \int_0^\infty \sum_{n=0}^\infty \frac{(-r)^n \mathrm{e}^{-xr}\mathrm{d}r}{n!\Gamma(1 - \alpha - \alpha n)}, \qquad 0 < \alpha < 1 \\
&= \frac{1}{2\pi i}\int_\Upsilon \mathrm{e}^{(r - xr^\alpha)} r^{\alpha-1}\mathrm{e}^{-xr}\mathrm{d}r, \qquad 0 < \alpha < 1
\end{aligned} \tag{10}$$

where Υ is a contour domain which encircles the origin counterclockwise, going from $-\infty$ to $+\infty$. Secondly, let us consider the definitions.

Definition 1. Maintaining the same notations as in (1), we consider an operator A applying in the fractional model:

$$D_t^{\alpha}[u(x,m,t)] = Au(x,m,t) \tag{11}$$

subject to the initial condition:

$$u(x,m,0) = f(x,m) \tag{12}$$

with $0 < \alpha < 1,\ t,\ m > 0,\ x \in \Omega \subset R^3$.

A family $[G_{\alpha}(t)]_{t>0}$ of bounded operators on H is called a solution operator of the fractional Cauchy problem (11)-(12) if:
- $G_{\alpha}(0) = I$
- $G_{\alpha}(t) =$ is strongly continuous for every $t \geq 0$
- $AG_{\alpha}(t)f = G_{\alpha}(t)Af$ for all $f \in D(A)$
- $G_{\alpha}(t)D(A) \subset D(A)$
- $G_{\alpha}(t)f$ is a (classical) solution of the model (11)-(12) for all $f \in D(A)$, $t \geq 0$.

It is well known [15, 25] that an operator $A \in G(M,\omega)$ means A generates a C_0 semi group $[G(t)]_{t>0}$ so that there exists $M > 0$ and ω such that:

$$\|G(t)\| \leq M\mathrm{e}^{\omega t} \tag{13}$$

In the same way, if the fractional Cauchy problem (11)-(12) has a solution operator $[G_{\alpha}(t)]_{t>0}$ verifying (13), then we say that $A \in G^{\alpha}(M,\omega)$. The solution operator $[G_{\alpha}(t)]_{t>0}$ is contractive if:

$$\|G_{\alpha}(t)\| \leq 1 \tag{14}$$

and we say $A \in G^{\alpha}(1, 0)$. Recall that if we have a contractive solution operator, we can use Definition 1 to identify the fractional Cauchy problem of which it is a solution. Usually, however, we are interested in the reverse question, that is, in finding the solution operator, that is contractive, for a given fractional model. The answer is given by the theorem (seen as an analogue of Hille-Yosida theorem).

Theorem 1. An operator $A \in G^{\alpha}(1,0)$ for $0 < \alpha \leq 1$ if and only if:

(a) $$(0,\infty) \subset \rho(A),$$

(b) $$\lambda^{\alpha-1}R(\lambda^{\alpha},A)f = \int_0^{\infty} \mathrm{e}^{-\lambda r}G(r)f\mathrm{d}r \tag{15}$$

with $\lambda > 0, f \in H$ where $[G(t)]_{t>0}$ is a family of strongly continuous operators satisfying (14) and $\rho(A)$ is the resolving set of the operator A:

$$\rho(A) = [\lambda \in R;\ \ \lambda I - A : D(A) \to H_1 \text{ is invertible and} (\lambda I - A)^{-1} \text{ bounded and linear}]. \tag{16}$$

Proof. This theorem is a particular version of [8, Theorem 1.3] and the proof follows the same steps.

Cauchy problem for diffusion-wave dynamics of fractional order

Our primary objective in this section is to show how a solution to the diffusion-wave equations can be approximated by a separation of variables.

$$D_t^{\alpha}[u(x,m,t)] = \nabla^2 u(x,m,t), \quad 0<\alpha<1, \quad t>0, \quad x>0; \quad u(x,m,0)=f(x), \quad x>0 \tag{17}$$

where ∇^2 is the Laplacian. To show existence result for this model, we use the separation of variables by setting $u(x,m,t) = X(x)T(t)$. Substitution in (17) gives:

$$X(x)D_t^{\alpha}T(t) = T(t)\nabla^2 X(x)$$

which can be re-arranged:

$$\frac{D_t^{\alpha}T(t)}{T(t)} = \frac{\nabla^2 X(x)}{X(x)}$$

We put $-\lambda = [D_t^{\alpha}T(t)]/T(t) = \nabla^2 X(x)/X(x)$ to get the eigenvalue system:

$$\nabla^2 X(x) = -\lambda X(x), \ x>0 \tag{18}$$

$$D_t^{\alpha}T(t) = -\lambda T(t), \ t>0 \tag{19}$$

To solve the eigenvalue system (18), we use an infinite sequence of pairs $\{\alpha_n,\delta_n\}_{n\in N}$ with $\{\alpha_n\}$ an increasing sequence such that $\alpha_n \to \infty$ and $\{\delta_n\}$ a family of functions that form a complete orthogonal set in $L^2(x_0,\infty)$. Exploiting α_n defined from (18), we can find a solution of the eigenvalue problem for the *α*-fractional derivative (19) by putting $\lambda = \alpha_n$ [26]. It is obvious that the Mittag-Leffler function:

$$E_\alpha\left(-\mu t^{\alpha}\right) = \sum_{n=0}^{\infty}\frac{(-\mu t^{\alpha})^n}{\Gamma(n\alpha+1)}. \tag{20}$$

is the unique solution of the eigenvalue problem:

$$D_t^{\alpha}T(t) = -\mu T(t), \ t>0, \quad T(0)=1 \tag{21}$$

Therefore the solution to (19) is given as:

$$T(t) = \tilde{f}(n)\sum_{n=0}^{\infty}\frac{(-\mu t^{\alpha})^n}{\Gamma(n\alpha+1)}$$

where $\tilde{f}(n)$ is chosen to satisfy the initial condition *f*. This leads us to a formal solution of the fractional Cauchy problem (17) given by:

$$u(x,m,t) = \sum_{n=1}^{\infty}\tilde{f}(n)E_\alpha(-\mu t^{\alpha})\delta_n(x) \tag{22}$$

Diffusion-wave dynamics of fractional order in *R*

Making use of the model (11) where the operator A is defined as:

$$Au = \mu\, u_{xx} \tag{23}$$

with $\mu \in R, \quad x \in R, \quad t > 0,$ we have the classical result that A generates an analytic semi-group of angle $\pi/2$. Moreover, for $\alpha = 2$, the operator is the generator of a strongly continuous semi-group $\left[G_2(t)\right]_{t>0}$ given by the d'Alembert formula:

$$[G_2(t)f](x) = \frac{1}{2}[f(x+t\sqrt{\mu}) + f(x-t\sqrt{\mu})] \tag{24}$$

Exploiting the subordination principle [1, 8], we obtain that there is a solution operator $\left[G_2(t)\right]_{t>0}$ for the model (11) generated by A defined in (23) such that:

$$G_\alpha(t)f = \int_0^\infty \Theta_\alpha(t,r)G_2(r)f\mathrm{d}r \tag{25}$$

where

$$\Theta_\alpha(t,r) = \frac{1}{t^\alpha}\sum_{n=0}^{\infty}\frac{(-r)^n t^{-\alpha n}}{n!\Gamma(1-\alpha-\alpha n)}$$

is the Wright type function [24, 27]. Therefore, this addresses the problem well-posedness for the model (11)-(12) since it admits a solution operator. This yields a classical solution of the system (11)-(12) given by:

$$u(x,m,t) = \int_0^\infty \Theta_\alpha(t,r)G_2(r)f\mathrm{d}r \qquad f \in D(A) \tag{26}$$

A kinetic-type fractional model with diffusion

The last model to investigate is the kinetic-type fractional model in $\Omega \times R_+$ with diffusion given as:

$$\begin{gathered} D_t^\alpha[u(x,m,\ t)] = D_x[d(x,m)D_x u(x,m,\ t)] \\ u(x,m,0) = f(x,m),\ x > 0 \\ \mathrm{D_N} u|_\Omega \ = 0 \end{gathered} \tag{27}$$

where $0 < \alpha < 1,\ t > 0,\ x \in \Omega \subseteq R^3$.

It is assumed that the diffusion coefficient d is bounded in the sense that there are two real constants δ_1 and δ_2 and a function $\theta \in L_{\infty,loc}(0,\infty)$ such that:

$$\delta_1\theta(m) \le d(x,m) \le \delta_2\theta(m). \tag{28}$$

Let us define the operator $[A, D(A)]$ by:

$$Au(x,m,t) = D_x[d(x,m)D_x u(x,m,t)]$$

acting in H according to the formulas:

$$D(A) := \{v \in H; v(m) \in L_{1,2}(R^3) \text{ for almost every } m,$$

$$(x,m) \to [A|_{L_{1,2}(R^3)} v(\cdot,m)](x) \in H\} \tag{29}$$

where $L_{1,2}(R^3)$ is the Bessel potential space as defined in [28]. Recall that in [29] for $m \in N$, $1 \le p \le \infty$, Sobolev spaces $W_p^m(\Omega)$ are given as:

$$W_p^m(\Omega) := \{v \in L_p(\Omega); \quad D^\beta v \in L_p(\Omega), \quad 0 \le |\beta| \le m\} \tag{30}$$

with $D^\beta = \partial_{\beta_1}/\partial_{x_1} \dots \partial_{\beta_n}/\partial_{x_n}$ and $|\beta| = \beta_1 + \cdots + \beta_n$ representing the distributional derivative of order $|\beta|$. The spaces $W_p^m(\Omega)$ are Banach spaces when endowed with the norm:

$$\|v\|_{W_p^m(\Omega)} = \left(\sum_{0 \le |\beta| \le m} \| D^\beta v \|_{L_p(\Omega)}^p \right)^{\frac{1}{p}}$$

To proceed, we need the following definitions.

Definition 1. (Bochner measurable function) [30]. Let $\Xi \subseteq R^n, n \in N$ be a measurable set with respect to the Lebesgue measure μ. A function v defined almost everywhere on Ξ is said to be Bochner measurable on Ξ if there exists a sequence $(v_n)_{n \in N}$ of simple functions such that:

$$\lim_{n \to \infty} \|v_n(t) - v(t)\| = 0$$

almost everywhere on Ξ.

Definition 2. (Stochastic and substochastic semigroups). The strongly continuous semi group of operators $(G(t))_{t \ge 0}$ on the Banach space $X = L_1(\Omega, \mu)$ is said to be:
– substochastic if $S(t) \ge 0$, and $\|G(t)\| \le 1$ for all $t \ge 0$;
– stochastic if, in addition, it satisfies $\|G(t)\psi\| \le |\,|\psi|\,|$ for all non-negative $\psi \in X$.

Now we can state the theorem 2.

Theorem 2. In addition to the mentioned conditions, if we assume that the diffusion coefficient $d(\cdot,m) \in W_\infty^1(\Omega)$ and is Bochner measurable then, there is a solution operator for the fractional model (27) that is positive and contractive.

Proof. To prove this theorem, we consider the model (27) with the derivative order $\alpha = 1$:

$$D_t^1[u(x,m,t)] = \frac{\partial}{\partial t} u(x,m,t) = D_x[d(x,m) D_x u(x,m,t)]$$

$$u(x,m,0) = f(x,m), \quad x > 0$$

$$D_N u|_\Omega = 0 \tag{31}$$

and set the operator $A_1 u(x,m,t) = D_x[d(x,m) D_x u(x,m,t)]$ with the same assumptions as mentioned. Then, making use of [31], the operator A_1 with the domain given by (29) is the generator of a stochastic semi group $[S(t)]_{t>0}$ in H exploiting the relaxation (10), let us set:

$$G_\alpha(t) f = \int_0^\infty \Theta_\alpha(t,r) G_K(r) f \mathrm{d}r \tag{32}$$

where $\Theta_\alpha(t,r)=\frac{1}{t^\alpha}\sum_{n=0}^{\infty}\frac{(-r)^n t^{-\alpha n}}{n!\Gamma(1-\alpha-\alpha n)}$

By the subordination principle [1], $(G_\alpha)_{t>0}$ a solution operator for the fractional model (fktgrowth7). It remains to prove that $(G_\alpha)_{t>0}$ is positive and contractive. The positivity of $(G_\alpha)_{t>0}$ follows from (25), the fact that $[S(t)]_{t>0}$ is positive together with the monotonicity of the Mittag-Leffler function $E_\alpha(-x)$ taken from (10). For the contractility of $(G_\alpha)_{t>0}$, recall that Hille-Yosida theorem implies $(0,\infty)\subset\rho(A_1)$ and since $S(t)$ is sub-stochastic, we have $\|S(r)\|\le 1,\ r\ge 0.$ Next, we exploit the fact that [5, 24]:

$$\int_0^\infty\left[\sum_{n=0}^{\infty}\frac{(-r)^n}{n!\Gamma(1-\alpha-\alpha n)}\right]\mathrm{d}r=1 \tag{33}$$

together with the second last relation in (10) to obtain:

$$\|G_\alpha(r)\|\le\int_0^\infty\Theta_\alpha(t,r)\|S(r)\|\mathrm{d}r\le 1,\quad t\ge 0$$

Thus, $[G_\alpha(t)]_{t>0}$ is contractive.

Fractional kinetic-type fragmentation equation with diffusion

After addressing the well-posedness for the diffusion process, we propose to investigate the model understood as the fractional generalization of the full model of fragmentation dynamics with diffusion and expressed as:

$$D_t^\alpha[u(x,m,t)]=D_x[d(x,m)D_x u(x,m,t)]-a(x,m)u(x,m,t)+$$
$$+\int_m^\infty a(x,y)b(m,y)u(x,y,t)\,\mathrm{d}y \tag{34}$$

subject to the initial conditions:

$$u(x,m,0)=f(x,m),\quad D_N u\,|_\Omega\ =0 \tag{35}$$

To obtain the abstract reformulation of (34)-(35) in $H,$ let us denote by $\mathcal{A}$ and $\mathcal{F}$ the expressions appearing on the RHS of (34); that is:

$$[\mathcal{A}\psi](x)=D_x[d(x,m)D_x(\psi(x)] \tag{36}$$

$$[\mathcal{F}\psi](x)=-a(x,m)\psi(x)+\int_m^\infty a(x,y)b(m,y)\psi(y)\mathrm{d}y \tag{37}$$

Thus we are led to the abstract form of (34)-(35):

$$D_t^\alpha[\mathrm{u(t)}]=[A+F]u(t),\ \ t>0,\quad u(0)=f,\quad D_N u\,|_\Omega\ =0 \tag{38}$$

where the operators A and F are the realizations of $\mathcal{A}$ and $\mathcal{F}$ on H, respectively. To go further, we need to mention that the operators on the RHS of (34) have the property that one of the variable can play the role of a parameter for which each fixed value provides the operator with certain useful properties (like generating a semi group) with respect to the other variable. Then they are parameter-dependent operators that can be rearranged together in such a way that the resulting operator inherits the properties of the individual components. Using this idea in the diffusion part of (34), the x variable is the parameter and m is the main variable. We can therefore construct the gain part and loss part of the fragmentation process, respectively, characterized by the operators B_x and E_x assumed to satisfy:

$$\int_0^\infty (E_x u + B_x u) m\mathrm{d}m = 0 \tag{39}$$

where

$$E_x u(x,m,\ t) = -a(x,\cdot)u(x,m,\ t), \quad u \in Hm = L_1(R, m\mathrm{d}m) \tag{40}$$

and

$$B_x u(x,m,\ t) = \int_m^\infty a(x,y)b(m,y)u(x,y,\ t)\mathrm{d}y, \quad u \in Hm = L_1(R, m\mathrm{d}m) \tag{41}$$

Assuming that there are two real constants a_1 and a_2, and a function $v \in L_{loc,\ \infty}(0,\infty)$ such that:

$$a_1(m) \le a(x,m) \le a_2(m) \tag{42}$$

Following the technique described, see also [31], we denote by [B, $D(B)$] and [E, $D(E)$], respectively, the operators in the whole space H issued from re-arrangement of the operators Bx and Ex, respectively, according to the eq. (29). It is obvious that $F = E + B$, and then, the condition (42) yields $D(F) = D(E) \cap D(B) = D(E)$.

Hence, all the necessary conditions have now been provided in order to prove the well-posedness of the fractional model (38) in H. To do so, we shall use the same approach as in Theorem 2.3, make use of [31, Theorem 2–b] together with the subordination principle from the equation:

$$D_t^1[u(x,m,\ t)] = \frac{\partial}{\partial t}u(x,m,\ t) = D_x[d(x,m)D_x u(x,m,\ t) - a(x,\ m)u(x,m,\ t)] + \\ + \int_m^\infty a(x,y)b(m,y)u(x,y,\ t)\mathrm{d}y \tag{43}$$

We have therefore proved the theorem 3.

Theorem 3. In addition to the conditions of Theorem 2, assume that (39) and (42) hold. For the fractional model (38), there an extension of the operator $A + F$ which is the generator of a positive and contractive solution operator.

Concluding remarks

We have used relatively new concepts like solution operators for evolution equations with derivative of fractional order, together with alternative methods like separation of variables and subordination principle to set up adequate assumptions allowing us to investigate and prove, in the space $L_1(R^3 \times R_+, m\mathrm{d}m\mathrm{d}x)$, the well-posedness of a fractional fragmentation process in a kinetic-diffusive-wave medium. This is the first instance where such a model is investigated and the work presented in this paper improves the preceding ones with the inclusion of fractional derivative, leading to models that have already been proved to be well defined [1, 5, 8] in a very large scale. In particular we proved that the solution operator for the model is positive and contractive. Therefore, the results obtained here may represent the first step of a wider work that will certainly contribute to the full explanation of strange phenomenon like the sudden appearance of an infinite number of particles in a systems which had an initial finite number of particles. The problem remains open.

Nomenclature

$a(x, m)$ – fragmentation rate
$b(s, m)$ – distribution kernel
$\mathrm{d}(x, m)$ – diffusion coefficient
m – mass of a particle
N – outward normal to the boundary $\partial\Omega$
t – time
$u(x, m, t)$ – mass-position density function
x – position of a particle

Greek symbols

Ω – open subset of R^3
$\partial\Omega$ – boundary

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