

## A NEW ITERATION ALGORITHM FOR SOLVING THE DIFFUSION PROBLEM IN NON-DIFFERENTIABLE HEAT TRANSFER

by

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*In the article, the variational iteration algorithm LFVIA-II is implemented to solve the diffusion equation occurring in non-differentiable heat transfer. The operators take in sense of the local fractional operators. The obtained results show the fractal behaviors of heat transfer with non-differentiability.*

*Key words: variational iteration algorithm, LFVIA-II, diffusion equation, heat transfer, non-differentiable, local fractional derivative*

### Introduction

Fractals are basic characteristics of nature of the fractal surfaces, patterns and structures [1-3] especially in the surface science, such as fracture surfaces of metals [4], natural scenes [5], super water-and oil-repellent surfaces [6], surface roughening of rocks [7] and so on (see also the references cited in each of these works). In fact, some rough surfaces of materials are characterized by fractal Cantor structures. For example, the fractal laminar heat transfer in microchannels was considered [7]. The researchers from many areas of sciences and technologies has applied the theory of local fractional calculus to deal with the some problems in fractal Cantor structures in the fields of mathematics, physics, and computer science [8-12]. The local fractional differential equations in mathematical physics were observed in different coordinates of Cantor-type [13]. In [14], the diffusion equations of fractal heat transfer with local fractional derivative were reported. Several approaches for solving partial differential equations defined on Cantor sets were studied, such as the LVI [15], LFLT [16], VI [17, 18], LFAD [19, 20], FD [21, 22], LFFS [23], and LFST [24] methods and so on. In this manuscript, VIA-II within local fractional operator [18, 25, 26] will be applied to deal with the diffusion problem in non-differentiable heat transfer.

### Analysis of the methodology

We now consider the diffusion equation for non-differentiable transfer in the form [8]:

$$L_{\alpha}T + R_{\alpha}T = 0 \quad (1)$$

where  $R_{\alpha} = D_{\alpha} \partial^{2\alpha} T(x,t) / \partial x^{2\alpha}$  and  $L_{\alpha} = \partial^{\alpha} T(x,t) / \partial t^{\alpha}$ , which is defined as [8-26]:

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$$\left. \frac{\partial^\alpha T(x, y)}{\partial x^\alpha} \right|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha [T(x, y) - T(x_0, y)]}{(x - x_0)^\alpha} \quad (2)$$

with  $\Delta^\alpha [T(x, y) - T(x_0, y)] \cong \Gamma(1 + \alpha) \Delta [T(x, y) - T(x_0, y)]$ .

Let us structure the correction local fractional functional given as:

$$T_{n+1}(x, t) = T_n(x, t) + {}_0 I_t^{(\alpha)} \{ \mu [L_\alpha T_n(x, \tau) + R_\alpha \tilde{T}_n(x, \tau)] \} \quad (3)$$

where  $\tilde{T}_n$  is the restricted local fractional variation,  $\mu$  – the fractal Lagrange multiplier, and the local fractional integral operator denotes [8, 18-26]:

$${}_\sigma I_\nu^{(\alpha)} g(s) = \frac{1}{\Gamma(1 + \alpha)} \int_\sigma^\nu g(s) (ds)^\alpha = \frac{1}{\Gamma(1 + \alpha)} \lim_{\Delta s \rightarrow 0} \sum_{j=0}^{N-1} f(s_j) (\Delta s_j)^\alpha \quad (4)$$

with  $\Delta s = \max \{ \Delta s_1, \Delta s_2, \Delta s_j, \dots \}$  and  $\Delta s_j = s_{j+1} - s_j$  ( $s_0 = \sigma, s_N = \nu$ ),  $j = 0, \dots, N - 1$ .

Using the local fractional variational principle [8], we obtain:

$$\mu = -1 \quad (5)$$

such that the LFDIA-II can be written as:

$$T_{n+1}(x, t) = T_0(x, t) - {}_0 I_t^{(\alpha)} \left[ D_\alpha \frac{\partial^{2\alpha} T_n(x, t)}{\partial x^{2\alpha}} \right] \quad (6)$$

where

$$T_0(x, t) = T(x, 0) \quad (7)$$

Finally, from eq. (6) the non-differentiable solution is given as:

$$T = \lim_{n \rightarrow \infty} T_n \quad (8)$$

### A solution to diffusion problem in non-differentiable heat transfer

When  $D = 2$ , we consider the diffusion model for the non-differentiable heat transfer:

$$\frac{\partial^\alpha T(x, t)}{\partial t^\alpha} - 2 \frac{\partial^{2\alpha} T(x, t)}{\partial x^{2\alpha}} = 0 \quad (9)$$

subject to initial-boundary conditions:

$$\frac{\partial^\alpha T(0, t)}{\partial x^\alpha} = 0, \quad T(x, 0) = E_\alpha(x^\alpha), \quad T(0, t) = 0 \quad (10)$$

Considering eq. (6), the following successive approximations can be found to be:

$$T_1(x, t) = T_0(x, t) - {}_0 I_t^{(\alpha)} \left[ 2 \frac{\partial^{2\alpha} T_0(x, t)}{\partial x^{2\alpha}} \right] = E_\alpha(x^\alpha) \left[ 1 - \frac{2t^\alpha}{\Gamma(1 + \alpha)} \right] \quad (11)$$

$$\begin{aligned} T_2(x, t) &= T_0(x, t) - {}_0 I_t^{(\alpha)} \left[ 2 \frac{\partial^{2\alpha} T_1(x, t)}{\partial x^{2\alpha}} \right] = \\ &= E_\alpha(x^\alpha) - {}_0 I_t^{(\alpha)} \left\{ 2 E_\alpha(x^\alpha) \left[ 1 - \frac{2t^\alpha}{\Gamma(1 + \alpha)} \right] \right\} = E_\alpha(x^\alpha) \sum_{i=0}^2 \frac{2^i (-1)^i t^{i\alpha}}{\Gamma(1 + \alpha)} \end{aligned} \quad (12)$$

$$\begin{aligned}
 T_3(x,t) &= T_0(x,t) - {}_0I_t^{(\alpha)} \left[ 2 \frac{\partial^{2\alpha} T_2(x,t)}{\partial x^{2\alpha}} \right] = \\
 &= E_\alpha(x^\alpha) - {}_0I_t^{(\alpha)} \left[ 2E_\alpha(x^\alpha) \sum_{i=0}^2 \frac{2^i (-1)^i t^{i\alpha}}{\Gamma(1+i\alpha)} \right] = E_\alpha(x^\alpha) \sum_{i=0}^3 \frac{2^i (-1)^i t^{i\alpha}}{\Gamma(1+i\alpha)} \quad (13)
 \end{aligned}$$

$$\begin{aligned}
 T_4(x,t) &= T_0(x,t) - {}_0I_t^{(\alpha)} \left[ 2 \frac{\partial^{2\alpha} T_3(x,t)}{\partial x^{2\alpha}} \right] = \\
 &= E_\alpha(x^\alpha) - {}_0I_t^{(\alpha)} \left[ 2E_\alpha(x^\alpha) \sum_{i=0}^3 \frac{2^i (-1)^i t^{i\alpha}}{\Gamma(1+i\alpha)} \right] = E_\alpha(x^\alpha) \sum_{i=0}^4 \frac{2^i (-1)^i t^{i\alpha}}{\Gamma(1+i\alpha)} \quad (14)
 \end{aligned}$$

⋮

$$T_n(x,t) = T_0(x,t) - {}_0I_t^{(\alpha)} \left[ 2 \frac{\partial^{2\alpha} T_{n-1}(x,t)}{\partial x^{2\alpha}} \right] = E_\alpha(x^\alpha) \sum_{i=0}^n \frac{2^i (-1)^i t^{i\alpha}}{\Gamma(1+i\alpha)} = E_\alpha(x^\alpha) E_\alpha(-2t^\alpha) \quad (15)$$

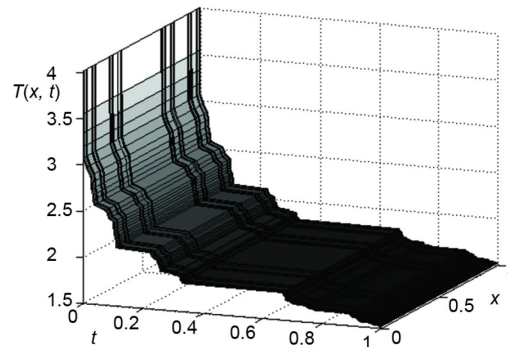
Hence, the non-differentiable solution of eq. (9) in the closed form is easily written as:

$$T(x,t) = \lim_{n \rightarrow \infty} T_n(x,t) = E_\alpha(x^\alpha) E_\alpha(-2t^\alpha) \quad (16)$$

and its chart is illustrated in fig. 1 when  $\alpha = \ln 2 / \ln 3$ .

**Conclusions**

The diffusion problem for the non-differentiable heat transfer had investigated in this manuscript. The LFDIA-II was utilized to find the solution for fractal diffusion equations in homogeneous media with local fractional derivatives. The obtained result with chart reveals the non-differentiable behaviors of the fractal heat transfer.



**Figure 1. The non-differentiable solution of eq. (9) in the closed form when  $\alpha = \ln 2 / \ln 3$**

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**Nomenclature**

$T(x, t)$  – concentration, [–]  
 $t$  – time, [s]  
 $x$  – space co-ordinate, [m]

*Greek symbol*  
 $\alpha$  – time fractal dimensional order, [–]

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