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SOLVING FRACTAL STEADY HEAT-TRANSFER PROBLEMS WITH THE LOCAL FRACTIONAL SUMUDU TRANSFORM

by

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In this paper the linear oscillator problem in fractal steady heat-transfer is studied within the local fractional theory. In particular, the local fractional Sumudu transform will be used to solve both the homogeneous and the non-homogeneous local fractional oscillator equations under fractal steady heat-transfer. It will be shown that the obtained non-differentiable solutions characterize the fractal phenomena with and without the driving force in fractal steady heat transfer at low excess temperatures.

Key words: *fractal heat transfer, oscillator equation, local fractional derivative, local fractional Sumudu transform*

Introduction

In view of complexity of the surfaces in solids, liquid and gas, local fractional calculus (LFC) was adopted to deal with some non-differentiable important problems both in applied and theoretical science, like *e. g.* heat transfer [1-3], oscillator motion of free damped vibrations [4] and others [5, 6]. In particular, the linear local fractional oscillator equations (LFOE) were recently developed to describe the fractal steady heat-transfer [7]. The local fractional Sumudu transform (LFST) was also developed to the local fractional ordinary differential equations (ODE) [8]. Thus, by combining these two models (LFOE) and the LFST we have the possibility to explore the oscillator equations arising in the heat transfer by using the LFST. In this paper our aim is to find the non-differentiable solutions (NS) for the linear LFOE using the LFST.

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Analytical solutions for the LFOE under fractal heat transfer

Let us consider the non-homogeneous LFOE in fractal heat transfer with an additional driving force $\omega(\mu)$ at low excess temperature [7]:

$$\nu \frac{\mathrm{d}^{2\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{2\vartheta}} + \kappa\varpi(\mu) = \omega(\mu) \tag{1a}$$

with the initial conditions:

$$\left.\frac{\mathrm{d}^{\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{\vartheta}}\right|_{\mu=\mu_0} = \varphi, \quad \varpi(\mu_0) = \psi \tag{1b,c}$$

being φ and ψ are two given constant values. In eq. (1a), ν and κ are two known parameters, and the local fractional derivative (LFD) of a given function $\mathrm{H}(\mu)$ is defined as [1-8]:

$$\left.\frac{\mathrm{d}^{\vartheta}\mathrm{H}(\mu)}{\mathrm{d}\mu^{\vartheta}}\right|_{\mu=\mu_0} = \lim_{\mu\to\mu_0} \frac{\Delta^{\vartheta}[\mathrm{H}(\mu) - \mathrm{H}(\mu_0)]}{(\mu - \mu_0)^{\vartheta}} \tag{1d}$$

with

$$\Delta^{\vartheta}[\mathrm{H}(\mu) - \mathrm{H}(\mu_0)] \cong \Gamma(1 + \vartheta)[\mathrm{H}(\mu) - \mathrm{H}(\mu_0)] \tag{1e}$$

The LFST via the Mittag-Leffler function on fractal sets [5] is defined as [8]:

$$LFS_{\vartheta}\{\psi(t)\} = \frac{1}{\Gamma(1 + \vartheta)} \int_0^{\infty} \frac{E_{\vartheta}(-h^{-\vartheta}t^{\vartheta})\psi}{h^{\vartheta}} (\mathrm{d}t)^{\vartheta} \tag{1f}$$

where the local fractional integral of $\psi(t)$ is [5, 8]:

$${}_{t_0}I_t^{(\vartheta)}\psi(t) = \frac{1}{\Gamma(1 + \vartheta)} \int_{t_0}^{t} \psi(t)(\mathrm{d}t)^{\vartheta} = \frac{1}{\Gamma(1 + \vartheta)} \lim_{\Delta t\to 0} \sum_{j=0}^{j=N-1} \psi(t_j)(\Delta t)^{\vartheta} \tag{1g}$$

with the partitions of the interval $[t_0, t]$ are sub-intervals (t_i, t_{i+1}), $j = 0, \ldots, N-1$, $\Delta t = t_{i+1} - t_i$.

According to this property of the LFST [8]:

$$LFS_{\vartheta} \frac{\mathrm{d}^{n\vartheta}\psi(t)}{\mathrm{d}t^{n\vartheta}} = \frac{1}{h^{n\vartheta}} \left\{ LFS_{\vartheta}\{\psi(t)\} - \sum_{k=0}^{n-1} h^{k\vartheta}\psi^{(k\vartheta)}(0) \right\} \tag{2a}$$

we have for the LFST of the derivative in eq. (1a):

$$LFS_{\vartheta} \frac{\mathrm{d}^{2\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{2\vartheta}} = \frac{1}{h^{2\vartheta}} (LFS_{\vartheta}\{\varpi(\mu)\} - \psi - h^{\vartheta}\varphi) \tag{2b}$$

so that:

$$\frac{\nu}{h^{2\vartheta}}(LFS_\vartheta\{\varpi(\mu)\}-\psi-h^\vartheta\varphi)+\kappa LFS_\vartheta\{\varpi(\mu)\}=LFS_\vartheta\{\omega(\mu)\} \tag{2c}$$

which leads us to:

$$LFS_\vartheta\{\varpi(\mu)\}=\frac{1}{1+\frac{\kappa}{\nu}h^{2\vartheta}}(\psi+h^\vartheta\varphi)+\frac{h^{2\vartheta}}{\nu}\frac{LFS_\vartheta\{\omega(\mu)\}}{1+\frac{\kappa}{\nu}h^{2\vartheta}} \tag{2d}$$

According to the local fractional convolution theorem and taking into account the properties of LFST [8], it is:

$$LFS_\vartheta\{\sin_\vartheta(at^\vartheta)\}=\frac{ah^\vartheta}{1+a^2h^{2\vartheta}},\qquad LFS_\vartheta\{\cos_\vartheta(at^\vartheta)\}=\frac{1}{1+a^2h^{2\vartheta}} \tag{3a,b}$$

from where we get the exact solution of the non-homogeneous LFOE in fractal heat transfer:

$$\varpi(\mu)=\psi\cos_\vartheta\left(\sqrt{\frac{\kappa}{\nu}}\mu^\vartheta\right)+\frac{\nu\varphi\sin_\vartheta\left(\sqrt{\frac{\kappa}{\nu}}\mu^\vartheta\right)+\omega(\mu)}{\kappa}-$$
$$-\frac{\int\limits_0^\mu\omega(t)\cos_\vartheta\left(\sqrt{\frac{\kappa}{\nu}}(\mu-t)^\vartheta\right)(\mathrm{d}t)^\vartheta}{\kappa\,\Gamma(1+\vartheta)} \tag{3c}$$

Discussion

In this section we will consider some special cases and we will analyze the solution (3c) under some additional conditions.

Let us first assume that the driving force term $\omega(\mu)=0$. The homogeneous LFOE in fractal heat transfer becomes, see also [7]:

$$\nu\frac{\mathrm{d}^{2\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{2\vartheta}}+\kappa\varpi(\mu)=0 \tag{4a}$$

subject to the initial conditions:

$$\frac{\mathrm{d}^\vartheta\varpi(\mu_0)}{\mathrm{d}\mu^\vartheta}=\varphi,\qquad\varpi(\mu_0)=\psi \tag{4b,c}$$

By virtue of eq. (3c) we obtain the exact solution of the non-differentiable type of the homogeneous LFOE (4a):

$$\varpi(\mu)=\psi\cos_\vartheta\left(\sqrt{\frac{\kappa}{\nu}}\mu^\vartheta\right)+\frac{\nu\varphi\sin_\vartheta\left(\sqrt{\frac{\kappa}{\nu}}\mu^\vartheta\right)}{\kappa} \tag{4d}$$

When $\omega(\mu)=1$, the solution (3c) becomes:

$$\varpi(\mu)=\psi\cos_\vartheta\left(\sqrt{\frac{\kappa}{\nu}}\mu^\vartheta\right)+\frac{\left(\nu\varphi-\sqrt{\frac{\kappa}{\nu}}\right)\sin_\vartheta\left(\sqrt{\frac{\kappa}{\nu}}\mu^\vartheta\right)+1}{\kappa} \tag{5a}$$

and it represents the exact solution of the non-homogeneous LFOE under fractal heat transfer:

$$\nu \frac{\mathrm{d}^{2\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{2\vartheta}} + \kappa\varpi(\mu) = 1 \tag{5b}$$

subject to the initial conditions:

$$\frac{\mathrm{d}^{\vartheta}\varpi(\mu_0)}{\mathrm{d}\mu^{\vartheta}} = \varphi, \qquad \varpi(\mu_0) = \psi \tag{5c,d}$$

Equations (4a) and (5b) describe the oscillator equations (OE) that arise in the homogeneous and non-homogeneous heat-transfer equations in fractal media via LFC [7].

Let us now consider some special values of the physical parameters and the rational orders (of derivatives).

When $\kappa = \nu$, $\psi = 1$, $\varphi = 1$, and $\vartheta = \ln2/\ln3$, the homogeneous LFOE in fractal heat transfer reads as:

$$\frac{\mathrm{d}^{2\vartheta}\varpi(\mu)}{\mathrm{d}\mu^{2\vartheta}} + \varpi(\mu) = 0 \tag{6a}$$

and the corresponding solution of eq. (6a) can be written in the form:

$$\varpi(\mu) = \cos_{\vartheta}(\mu^{\vartheta}) + \sin_{\vartheta}(\mu^{\vartheta}) \tag{6b}$$

When $\kappa = \nu$, $\psi = 1$, $\varphi = 1$, and $\vartheta = 1$, we simply obtain the homogeneous OE in heat transfer via conventional (integer) derivative (CD):

$$\frac{\mathrm{d}^{2}\varpi(\mu)}{\mathrm{d}\mu^{2}} + \varpi(\mu) = 0 \tag{7a}$$

and the corresponding solution of eq. (7a) can be simply written as:

$$\varpi(\mu) = \cos\mu + \sin\mu \tag{7b}$$

If we compare the differentiable solutions (7b) and non-differentiable (6b) for the conventional homogeneous OE and the homogeneous LFOE, respectively, we have the two represented in fig. 1, which coincide at the limit value $\vartheta = 1$. For $0 < \vartheta < 1$ instead, the plot of the solution, corresponding to the non-integer derivative, looks like a symmetric Cantor function.

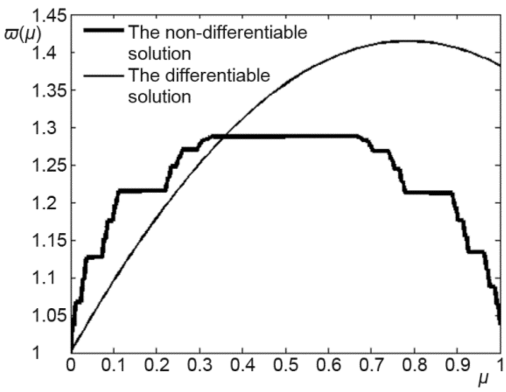

Figure 1. The comparison for the solutions for the homogeneous OE based on the LFD and the CD when $\kappa = \nu$, $\psi = 1$, and $\varphi = 1$

Conclusion

In this work the linear oscillator problems in fractal steady heat-transfer via the LFD were investigated. The exact solutions for the homogeneous and non-homogeneous LFOE in fractal heat transfer were also presented. The comparison between the homogeneous OE via the LFD and the CD is also discussed. The mathematical models for the linear LFOE efficiently characterize the fractal phenomena with and without the driving force in fractal steady heat transfer at low excess temperatures. The models of the linear oscillator problems are also extended using the new derivative [5, 9, 10].

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Nomenclature

$\mathrm{d}^{\vartheta}/\mathrm{d}\mu^{\vartheta}$ – LFD of ϑ, [–]
${}_{t_0}I_t^{(\vartheta)}$ – LFIO, [–]
LFS_{ϑ} – LFST, [–]

Greek symbols

μ – space co-ordinate, [m]
ϑ – fractal dimensional order, [–]
$\varpi(\mu)$ – temperature field, [K]

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