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ON THE FRACTAL HEAT TRANSFER PROBLEMS WITH LOCAL FRACTIONAL CALCULUS

by

Duan ZHAO [a,b]***, Xiao-Jun YANG*** [c]***, and Hari M. SRIVASTAVA*** [d*]

[a] IOT Perception Mine Research Center, China University of Mining and Technology, Xuzhou, China
[b] The National and Local Joint Engineering Laboratory of Internet Application Technology on Mine, Xuzhou, China
[c] Department of Mathematics and Mechanics, China University of Mining and Technology, Xuzhou, China
[d] Department of Mathematics and Statistics, University of Victoria, Victoria, B. C., Canada

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This article presents the fractal heat transfer problems from the local fractional calculus point of view. At low and high excess temperatures, the linear and non-linear heat transfer equations are presented. The non-homogeneous linear and non-linear oscillator equations in fractal heat transfer are discussed. The results are adopted to present the behaviors of the heat transfer in fractal media.

Key words: *fractal heat transfer equation, low excess temperature, high excess temperature, oscillator equation, local fractional calculus*

Introduction

Fractional calculus (FC) has potential applications in applied science and engineering practice with the help of kernel functions that are differentiable [1-8] or non-differentiable [9-11]. The local fractional calculus (LFC) is adopted to describe the non-differentiable problems in physics, such as Burgers' equation (BE) [12], parabolic Fokker-Planck equation (PFPE) [13], oscillator equation (OE) [14], diffusion equation (DE) [15-18], wave equation (WE) [19-21], Laplace equation (LE) [22], signal processing [23], and others [24, 25].

Based upon the results, we plan to structure the fractal heat transfer equations in fractal media at low and high excess temperatures. In this communication, our main aim is to propose the linear and non-linear heat transfer equations from the local fractional calculus point of view and to present the linear and non-linear oscillator equations arising in fractal heat transfer.

The non-homogeneous heat equation in fractal media

The non-homogeneous heat equation in fractal 3ω-dimensional space is written in the form [9-11]:

* Corresponding author; e-mail: dyangxiaojun@163.com

$$\mu^{2\omega}\nabla^{2\omega}\Theta(\theta, \vartheta, \phi, \tau) + \Phi(\theta, \vartheta, \phi, \tau) - \rho_\omega c_\omega \frac{\partial^\omega \Theta(\theta, \vartheta, \phi, \tau)}{\partial \tau^\omega} = 0 \tag{1}$$

where $\nabla^{2\omega} = \nabla^\omega \nabla^\omega$ [9-11], $\mu^{2\omega}$ is the heat conductivity of fractal materials, ρ_ω – the density of fractal materials, c_ω – the specific heat of fractal materials, $\Theta(\theta, \vartheta, \phi, \tau)$ – the temperature of fractal materials, and $\Phi(\theta, \vartheta, \phi, \tau)$ – the energy generation of fractal materials.

The non-homogeneous heat equation in fractal ω-dimensional space reads [9-11]:

$$\rho_\omega c_\omega \frac{\partial^\omega \Theta(\phi, \tau)}{\partial \tau^\omega} - \mu^{2\omega} \frac{\partial^{2\omega} \Theta(\phi, \tau)}{\partial \phi^{2\omega}} - \Phi(\phi, \tau) = 0 \tag{2a}$$

where $\mu^{2\omega}$ is the heat conductivity of fractal materials, ρ_ω – the density of fractal materials, c_ω – the specific heat of fractal materials, $\Theta(\phi, \tau)$ – the temperature of fractal materials, and $\Phi(\phi, \tau)$ – the energy generation of fractal materials.

The initial-boundary conditions of eq. (2a) are:

$$\frac{\partial^\omega \Theta(0, \tau)}{\partial \phi^\omega} = \varphi_\vartheta(\tau), \qquad \Theta(\phi, 0) = \psi(\phi) \tag{2b,c}$$

The linear heat transfer equations in fractal media

At low excess temperatures, accounting for the radiative loss of heat $\rho_\omega c_\omega \varpi\, \Theta(\phi, \tau)$, the heat transfer equation in fractal media is presented:

$$\rho_\omega c_\omega \frac{\partial^\omega \Theta(\phi, \tau)}{\partial \tau^\omega} + \mu^{2\omega} \frac{\partial^{2\omega} \Theta(\phi, \tau)}{\partial \phi^{2\omega}} + \rho_\omega c_\omega \varpi\, \Theta(\phi, \tau) - \Phi(\phi, \tau) = 0 \tag{3a}$$

which leads to:

$$\frac{\partial^\omega \Theta(\phi, \tau)}{\partial \tau^\omega} - \frac{\Phi(\phi, \tau)}{\rho_\omega c_\omega} + \frac{\mu^{2\omega}}{\rho_\omega c_\omega} \frac{\partial^{2\omega} \Theta(\phi, \tau)}{\partial \phi^{2\omega}} + \varpi\, \Theta(\phi, \tau) = 0 \tag{3b}$$

Setting $\Phi_\omega(\phi, \tau) = \Phi(\phi, \tau)/\rho_\omega c_\omega$ and $\kappa = \mu^{2\omega}/\rho_\omega c_\omega$, eq. (3b) is rewritten:

$$\frac{\partial^\omega \Theta(\phi, \tau)}{\partial \tau^\omega} + \kappa \frac{\partial^{2\omega} \Theta(\phi, \tau)}{\partial \phi^{2\omega}} + \varpi \Theta(\phi, \tau) = \Phi_\omega(\phi, \tau) \tag{3c}$$

subject to the initial-boundary conditions:

$$\frac{\partial^\omega \Theta(0, \tau)}{\partial \phi^\omega} = \varphi_\phi(\tau), \qquad \Theta(\phi, 0) = \psi(\phi) \tag{3d,e}$$

Equation (3c) is the non-homogeneous heat transfer equation in fractal media.

When $\Phi_\omega(\phi, \tau) = 0$, the homogeneous heat transfer equation in fractal media is written in the form:

$$\frac{\partial^\omega \Theta(\phi, \tau)}{\partial \tau^\omega} + \kappa \frac{\partial^{2\omega} \Theta(\phi, \tau)}{\partial \phi^{2\omega}} + \varpi\, \Theta(\phi, \tau) = 0 \tag{4a}$$

subject to the initial-boundary conditions:

$$\frac{\partial^{\omega}\Theta(0,\tau)}{\partial\phi^{\omega}}=\varphi_{\phi}(\tau),\qquad \Theta(\phi,0)=\psi(\phi) \tag{4b,c}$$

In view of eq. (4a), we obtain the steady non-homogeneous heat transfer equation (the oscillator equation with the driving force in fractal heat transfer) in fractal media:

$$\kappa\frac{\mathrm{d}^{2\omega}\Theta(\phi)}{\mathrm{d}\phi^{2\omega}}+\varpi\,\Theta(\phi)=\Phi_{\omega}(\phi) \tag{5}$$

where $\Phi_{\omega}(\phi)$ is the driving force term.

With eq. (5), the steady homogeneous heat transfer equation (the oscillator equation without the driving force in fractal heat transfer) in fractal media reads:

$$\kappa\frac{\mathrm{d}^{2\omega}\Theta(\phi)}{\mathrm{d}\phi^{2\omega}}+\varpi\,\Theta(\phi)=0 \tag{6}$$

The non-linear heat transfer equations in fractal media

At high excess temperatures, the radiative heat-loss given by $\rho_{\omega}c_{\omega}\varpi\,\Theta^4(\phi,\tau)$, we obtain the non-linear heat transfer equation in fractal media:

$$\frac{\partial^{\omega}\Theta(\phi,\tau)}{\partial\tau^{\omega}}+\kappa\frac{\partial^{2\omega}\Theta(\phi,\tau)}{\partial\phi^{2\omega}}+\xi\Theta^4(\phi,\tau)=\Phi_{\omega}(\phi,\tau) \tag{7a}$$

subject to the initial-boundary conditions:

$$\frac{\partial^{\omega}\Theta(0,\tau)}{\partial\phi^{\omega}}=\varphi_{\phi}(\tau),\qquad \Theta(\phi,0)=\psi(\phi) \tag{7b,c}$$

where $\xi=\varepsilon\varsigma\gamma/S\rho_{\omega}c_{\omega}$. Here, ς is the Stefan's constant (SC), ε – the characteristic constant (CC) of fractal material, γ – the sectional perimeter (SP) of the fractal bar, and S – the cross-sectional area (CA) of the fractal bar.

Equation (7a) is non-homogeneous non-linear heat transfer equation in fractal media.

When $\Phi_{\omega}(\phi,\tau)=0$, eq. (7a) is rewritten:

$$\frac{\partial^{\omega}\Theta(\phi,\tau)}{\partial\tau^{\omega}}+\kappa\frac{\partial^{2\omega}\Theta(\phi,\tau)}{\partial\phi^{2\omega}}+\xi\Theta^4(\phi,\tau)=0 \tag{8a}$$

subject to the initial-boundary conditions:

$$\frac{\partial^{\omega}\Theta(0,\tau)}{\partial\phi^{\omega}}=\varphi_{\phi}(\tau),\qquad \Theta(\phi,0)=\psi(\phi) \tag{8b,c}$$

where ξ is a constant. Equation (8a) is the homogeneous non-linear heat transfer equation in fractal media.

The steady non-homogeneous non-linear heat transfer equation (the non-linear oscillator equation with driving force) in fractal media is:

$$\kappa \frac{\mathrm{d}^{2\omega}\Theta(\phi)}{\mathrm{d}\phi^{2\omega}} + \xi\Theta^4(\phi) = \Phi_\omega(\phi) \tag{9}$$

where $\Phi_\omega(\phi)$ is the driving force term.

The steady homogeneous non-linear heat transfer equation in fractal media (the non--linear oscillator equation without driving force) is presented:

$$\kappa \frac{\mathrm{d}^{2\omega}\Theta(\phi)}{\mathrm{d}\phi^{2\omega}} + \xi\Theta^4(\phi) = 0 \tag{10}$$

Conclusions

In this work the linear and non-linear heat transfer equations via LFC are presented. Meanwhile, the linear and non-linear oscillator equations in fractal heat transfer are discussed. This result opens a new direction in fractal heat transfer.

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Nomenclature

θ, ϑ, ϕ – space co-ordinates, [m]
τ – time, [s]
$\Theta(\phi, \tau)$ – temperature, [K]
ω – time fractal dimensional order, [–]
$\partial^\omega/\partial\tau^\omega$ – local fractional partial derivative, [–]
$\mathrm{d}^\omega/\mathrm{d}\phi^\omega$ – local fractional derivative, [–]

References

[1] Kilbas, A. A., *et al.*, *Theory and Applications of Fractional Differential Equations,* North-Holland Mathematics Studies 204 (Ed. Jan van Mill), Elsevier, 2006
[2] Tarasov, V. E., Fractal Electrodynamics via Non-Integer Dimensional Space Approach, *Phys. Lett. A, 379* (2015), 36, pp. 2055-2061
[3] Hristov J., An Approximate Analytical (Integral-Balance) Solution to a Non-Linear Heat Diffusion Equation, *Thermal Science, 19* (2015), 2, pp. 723-733
[4] West, B. J., *et al.*, Fractional Calculus Ties the Microscopic and Macroscopic Scales of Complex Network Dynamics, *New J. Phys., 17* (2015), 4, 045009
[5] Machado, J. T., Mata, M. E., Pseudo Phase Plane and Fractional Calculus Modeling of Western Global Economic Downturn, *Commun. Non-linear Sci., 22* (2015), 1, pp. 396-406
[6] Bhrawy, A. H., *et al.*, A New Numerical Technique for Solving Fractional Sub-Diffusion and Reaction Sub-Diffusion Equations with a Non-Linear Source Term, *Thermal Science, 19* (2015), Suppl. 1, pp. S25-S34
[7] Chen, J., *et al.*, Analytical Solution for the Time-Fractional Telegraph Equation by the Method of Separating Variables, *J. Math. Anal. Appl., 338* (2008), 2, pp. 1364-1377
[8] Sandev, T., *et al.*, Fractional Diffusion Equation with a Generalized Riemann-Liouville Time Fractional Derivative, *J. Phys. A, 44* (2011), 25, 255203
[9] Yang. X. J., *et al.*, *Local Fractional Integral Transforms and Their Applications,* Elsevier, UK, 2015
[10] Cattani, C., *et al.*, *Fractional Dynamics,* Emerging Science Publishers, Berlin, 2015
[11] Yang, X. J., *Advanced Local Fractional Calculus and Its Applications,* World Science, New York, USA, 2012
[12] Yang, X. J., *et al.*, Nonlinear Dynamics for Local Fractional Burgers' Equation Arising in Fractal Flow, *Nonlinear Dynam.,* DOI 10.1007/S1071-015-2085-2

[13] Baleanu, D., *et al.*, Local Fractional Variational Iteration Algorithms for the Parabolic Fokker-Planck Equation Defined on Cantor Sets, *Progr. Fract. Differ. Appl., 1* (2015), 1, pp. 1-11
[14] Yang, X. J., Srivastava, H. M., An Asymptotic Perturbation Solution for a Linear Oscillator of Free Damped Vibrations in Fractal Medium Described by Local Fractional Derivatives, *Commun. Nonlinear Sci., 29* (2015), 1, pp. 499-504
[15] Yang, X. J., *et al.*, Local Fractional Similarity Solution for the Diffusion Equation Defined on Cantor Sets, *Appl. Math. Lett., 47* (2015), Sept., pp. 54-60
[16] Fan, Z. P., *et al.*, Adomian Decomposition Method for Three-Dimensional Diffusion Model in Fractal Heat Transfer Involving Local Fractional Derivatives, *Thermal Science, 19* (2015), Suppl. 1, pp. S137-S141
[17] Yang, X. J., *et al.*, Observing Diffusion Problems Defined on Cantor Sets in Different Coordinate Systems, *Thermal Science, 19* (2015), Suppl. 1, pp. S151-S156
[18] Jafari, H., *et al.*, A Decomposition Method for Solving Diffusion Equations via Local Fractional Time Derivative, *Thermal Science, 19* (2015), Suppl. 1, pp. S123-S129
[19] Yang, X. J., *et al.*, Local Fractional Homotopy Perturbation Method for Solving Fractal Partial Differential Equations Arising in Mathematical Physics, *Rom. Report. Phys., 67* (2015), 3, pp. 752-761
[20] Yan, S. P., *et al.*, Local Fractional Adomian Decomposition and Function Decomposition Methods for Laplace Equation within Local Fractional Operators, *Adv. Math. Phys., 2014* (2014), 161580
[21] Ahmad, J. S., Mohyud-Din, T., Solving Wave and Diffusion Equations on Cantor Sets, *Proc. Pakistan Acad. Sci., 52* (2015), 1, pp. 71-77
[22] Jassim, H. K., *et al.*, Local Fractional Laplace Variational Iteration Method for Solving Diffusion and Wave Equations on Cantor Sets within Local Fractional Operators, *Math. Probl. Eng., 2015* (2015), 309870
[23] Chen, Z. Y., *et al.*, Signal Processing for Nondifferentiable Data Defined on Cantor Sets: a Local Fractional Fourier Series Approach, *Adv. Math. Phys., 2014* (2014), 561434
[24] He, J.-H. A Tutorial Review on Fractal Spacetime and Fractional Calculus, *Int. J. Theor. Phys., 53* (2014), 11, pp. 3698-3718
[25] Liu, H. Y., *et al.*, Fractional Calculus for Nanoscale Flow and Heat Transfer, *Int. J. Numer. Method H., 24* (2014), 6, pp. 1227-1250

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