KINK DEGENERACY AND ROGUE WAVE 
FOR POTENTIAL KADOMTSEV-PETVIASHVILI EQUATION 

by 

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A new method called homoclinic breather limit method is proposed to solve the Potential Kadomtsev-Petviashvili equation, breather kink-wave and periodic soliton are obtained, and kink degeneracy and new rogue wave are first found in this paper. 

Key words: Potential Kadomtsev-Petviashvili equation, rogue wave solution, extended homoclinic test method, kink degeneracy, exp-function method 

Introduction 

In recent years, the study of exact solutions for the non-linear evolution equations has attracted much attention [1], and many non-linear phenomena were discovered [2-7]. 

In this work, we focus on some new non-linear phenomena, i. e., the kink degeneracy and rational breather solutions, for the (2+1) dimensional Potential Kadomtsev-Petviashvili (PKP) equation: 

$$u_{xt} + 6u_x u_{xx} + u_{xxxx} + u_{yy} = 0$$ (1) 

It is well-known that this equation arises in a number of remarkable non-linear problems in fluid mechanics and thermal science, and its solutions have been studied extensively. Solitary wave, soliton-like solution, and interaction among solitary waves were found [8-13]. Recently, exact periodic kink-wave and degenerative soliton solution of PKP were also elucidated [14]. In this paper, a novel approach named as homoclinic breather limit process is proposed to seek for rational breather-wave solutions. 

Homoclinic breather limit method 

In order to elucidate the new method, we consider a high dimensional non-linear evolution equation in the general form: 

$$F(u, u_t, u_x, u_y, u_{xx}, u_{yy}, \ldots) = 0$$ (2) 

where $u(x, y, t)$, and $F$ is a polynomial of $u$ and its derivatives. 

The new method takes following steps.

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Step 1. By Painleve analysis, a transformation:
\[ u = T(f) \]  
\[(3)\]
is made for a new and unknown function \( f \).

Step 2. Convert eq. (2) into Hirota bilinear form:
\[ G(D_x, D_y, f, f) = 0 \]  
\[(4)\]
where \( D \) is the operator defined by:
\[ D^m a b = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^k a(x, t) b(x', t') \bigg|_{x=x', t=t'} \]

Step 3. Solve eq. (3) using homoclinic test approach by assuming that [15]:
\[ f(x, y, t) = 1 + b_0 (e^{i\eta(x-at)} + e^{-i\eta(x-at)})e^{\Omega y + \gamma} + b_1 e^{2\Omega y + 2\gamma} \]  
\[(5)\]

or by the extended homoclinic test approach by assuming that:
\[ f(x, y, t) = e^{-p(\alpha x + \beta y + \alpha t + \gamma)} + b_0 \cos[p_1(\alpha_1 x + \beta_1 y + \alpha_1 t + \gamma_1)] + b_1 e^{p(\alpha x + \beta y + \alpha t + \gamma)} \]  
\[(6)\]
where \( \alpha, \beta, \Omega, \alpha_1, \beta_1, \omega_1, p, p_1, b_0, b_1, \gamma_1 \) are constants to be determined, or by a more general approach by the exp-function method [16].

Step 4. Substitute eq. (5) or eq. (6) into eq. (3), and equate all coefficients of 
\[ e^{-p(\alpha x + \beta y + \alpha t + \gamma)}, e^{\alpha x + \beta y + \alpha t + \gamma}, \cos(\alpha x + \beta y + \alpha t + \gamma), \sin(\alpha x + \beta y + \alpha t + \gamma), \] and constant term to zero, we obtain the set of algebraic equation for \( \alpha, \beta, \omega, \alpha_1, \beta_1, \omega_1, p, p_1, b_0, \) and \( b_1 \).

Step 5. Solve the set of algebraic equations in Step 4 using Maple and for \( \alpha, \beta, \Omega, \alpha_1, \beta_1, \omega_1, p, p_1, b_0, \) and \( b_1 \).

Step 6. Substitute the identified values of \( \alpha, \beta, \omega, \alpha_1, \beta_1, \omega_1, p, p_1, b_0, \) and \( b_1 \) into eq. (2) and eq. (5) and deduce the exact solutions of eq. (1).

Step 7. Using the relationship between \( p_1 \) and \( p \), or \( p \) and \( \Omega \) of exact solution obtained in the Step 6 and let \( p \) tends to zero, we can get the rational breather-wave solution.

Kink degeneracy

By using Painleve test we can assume that:
\[ u(x, y, t) = 2(\ln f)_x \]  
\[(7)\]
where \( f(x, y, t) \) is an unknown real function to be determined. Substituting eq. (7) into eq. (1), we obtain the bilinear equation:
\[ (D_x D_y + D_y^2 + D_x^2) f \cdot f = 0 \]  
\[(8)\]
Now we suppose that the solution of eq. (8) is:
\[ f(x, y, t) = e^{-\xi} + b_0 \cos(\eta) + b_1 e^{\xi} \]  
\[(9)\]
where
\[ \xi = p(x + \beta t + \omega), \quad \eta = p_1(x - \beta t + \omega) + \alpha_1 y \]
and \( \beta, \omega, \alpha_1, \omega_1, p, p_1, b_0, \) and \( b_1 \) are some constants to be determined later. Substituting eq. (9) into eq. (8), we have:
\[
\begin{align*}
2b_0\beta_1(p^4 + p_1^2 \beta + p^2 \beta + p_1^3 - 6p^2 p_1^2) \cos(\eta) - 8pb_0\beta_1(p - p_1)(p + p_1) \sin(\eta) \exp(\xi) + \\
+ [2b_0(p^4 + p_1^2 \beta + p^2 \beta + p_1^3 - 6p^2 p_1^2) \cos(\eta) + 8pb_0\beta_1(p - p_1)(p + p_1) \sin(\eta)] \exp(-\xi) - \\
- 2b_0^2 \alpha_1^2 + 8p^2 p_1^2 + 8b_0^2 p_1^4 + 2\beta b_0^2 p_1^2 + 32p^4 b_1 = 0
\end{align*}
\]
Equating all coefficients of different powers of \( \exp(\xi), \exp(-\xi), \sin(\eta), \) and \( \cos(\eta) \) to zero, we get:
\[
\begin{align*}
p^4 + p_1^2 \beta + p^2 \beta + p_1^3 - 6p^2 p_1^2 &= 0 \\
8pb_0\beta_1(p - p_1)(p + p_1) &= 0 \\
- 2b_0^2 \alpha_1^2 + 8p^2 p_1^2 + 8b_0^2 p_1^4 + 2\beta b_0^2 p_1^2 + 32p^4 b_1 &= 0
\end{align*}
\]
Solving the system of eqs. (10) with the aid of Maple, we get:
\[
\begin{align*}
p_1 &= \pm p, \\
b_1 &= \frac{b_0^2(\beta - 8p^2)}{4(\beta + 4p^2)}, \\
\alpha_1 &= \pm p\sqrt{2\beta - 4p^2}
\end{align*}
\]
and \( \beta, \omega, \omega_1, p, b_0, \) are free. Substitute eq. (11) into eq. (9), we obtain the solution:
\[
f(x, y, t) = \exp(-p(x + \beta t + \omega)) + b_0 \cos(p_1(x - \beta t + \omega_1) + \alpha_1 y) + b_1 \exp(p(x + \beta t + \omega))
\]
If \( 0 < b_1 \in R \), then we obtain the exact breather kink solution:
\[
u(x, y, t) = \frac{4p\sqrt{b_1} \sinh(p(x + \beta t + \omega)) + \ln(b_1) - 2b_0 \sin(p_1(x - \beta t + \omega_1) + \alpha_1 y)}{2\sqrt{b_1} \cosh(p(x + \beta t + \omega)) + \ln(b_1) + b_0 \cos(p_1(x - \beta t + \omega_1) + \alpha_1 y)}
\]
Solution \( u(x, y, t) \) represented by eq. (14) is a breather kink-wave solution. In fact, solution \( u(x, y, t) \) is a kink wave as trajectory along the straight line \( x = \beta t - \alpha_0 \), and meanwhile evolves periodically along the straight line \( x = -\beta t - \omega \), fig. 1(a).
Let \( p \) tends to zero and take \( b_0 = \pm 2 \), we can get the following rational breather solution:
\[
u(x, y, t) = \frac{-4\beta/(2x + \omega + \omega_1 + \sqrt{2\beta} y)}{A + B - 2\beta^3 t^2 - (2\omega \beta^2 + 2\beta^2 \alpha_1) t + 12 - \beta \omega^2 - \beta \omega_1^2}
\]
where
\[
\begin{align*}
A &= -2\beta x^2 + (2\beta \omega - 2\beta \omega_1 - 2\beta \sqrt{2} y) x \\
B &= -2\beta^2 y^2 + (2\beta \sqrt{2} t - 2\beta \sqrt{2} \omega_1 y)
\end{align*}
\]
The solution \( u(x, y, t) \) represented by eq. (15) is a breather solution and no longer the kink. This shows that kink is degenerated when the period of breather wave tends to infinite in the breather kink-wave, fig. 1(b). this is a new non-linear phenomenon up to now.

![Figure 1](image_url)

**Figure 1.** (a) the breather kink solution as \( p = 1/10, \beta = -2, b_0 = \omega_1 = \omega = 1, y = 0, \) and (b) the rational breather-wave solution as \( \alpha = \omega_1 = \omega = 0, \beta = 4, y = 0 \)

Solutions (13) is kinky periodic-wave which has speed \( \beta \), the forward-direction (or backward-direction) wave shows solitary feature meanwhile takes on kinky feature with space variable \( x, t \) for PKP equation. Specially, this wave shows both kinky and periodic feature to space variable \( t \). Such a surprising feature of weakly dispersive long-wave is first obtained. Meanwhile, notice (15) when the \( \beta \) tends to infinity, \( u \) tends to zero. From fig. 1, it is observed that the kink of the solutions disappeared when the \( p \) tends to zero. More importantly, we obtained a rational breather wave solution.

**From periodic soliton solution to rational breather wave solution**

In this section, by choosing special test function in application of HTA to (2+1) dimensional Potential-Petviashvili equation, we obtain a periodic soliton solution and a rational breather-wave solution. Setting:

\[
\xi = x - \alpha t
\]

where \( \alpha \) is a wave velocity. Equation (1) can be re-written as:

\[
-\alpha u_{\xi\xi} + 6u_{\xi}u_{\xi\xi} + u_{\xi\xi\xi\xi} + u_{\xi\xi} = 0
\]

Using Painlevé analysis, we suppose that the solution of eq. (11) is:

\[
u(x, y, t) = 2(\ln \ f)_\xi
\]

for some unknown real function \( f(x, y, t) \) and by substituting eq. (11) into eq. (12), we can obtain the bilinear form:

\[
(-\alpha D_\xi^2 + D_y^2 + D_{\xi\xi}^2)f \cdot f = 0
\]
With regard to eq. (13), using the homoclinic test technique [14], we are going to seek the solution of the form:

\[ f(\xi, y) = 1 + b_1 (e^{ip_1 \xi} + e^{-ip_1 \xi}) e^{\Omega y + \gamma} + b_2 e^{2\Omega y + 2\gamma} \]  

(19)

where \( p, \Omega, \gamma, b_1, \) and \( b_2 \) are all real to be determined below. Substituting eq. (14) into eq. (13) yields the exact solution of eq. (12) in the form:

\[ u(x, y, t) = \frac{-2b_1 p e^{\Omega y + \lambda} \sin(p_1 \xi)}{1 + 2b_1 \cos(p_1 \xi) e^{\Omega y + \gamma} + b_2 e^{2\Omega y + \gamma}} \]  

(20)

Computing \( D_\xi^2 f \cdot f, \quad D_y^2 f \cdot f, \) and \( D_{\xi y}^2 f \cdot f, \) we obtain:

\[
\begin{align*}
D_\xi^2 f \cdot f &= -4b_1 p^2 e^{\Omega y + \gamma} [\cos(p_1 \xi) e^{2\Omega y + 2\gamma} b_2 + \cos(p_1 \xi) + 2b_2 e^{\Omega y + \lambda}] \\
D_y^2 f \cdot f &= 4\Omega^2 e^{\Omega y + \gamma} [\cos(p_1 \xi) b_2 e^{2\Omega y + 2\gamma} b_2 + \cos(p_1 \xi) b_1 + 2b_2 e^{\Omega y + \lambda}] \\
D_{\xi y}^2 f \cdot f &= 4b_1 p^4 e^{\Omega y + \gamma} [\cos(p_1 \xi) e^{2\Omega y + 2\gamma} b_2 + \cos(p_1 \xi) + 8b_1 e^{\Omega y + \lambda}] 
\end{align*}
\]

(21)

Substituting eqs. (16) into eq. (13) we get:

\[
\begin{align*}
[(4b_1 p^2 \alpha b_2 + 4\Omega^2 b_2 + 4b_1 p^4 b_2) \cos(p_1 \xi) e^{2\Omega y + 2\gamma} + (4b_1 p^2 \alpha + 4\Omega^2 b_1 + 4b_1 p^4) \cos(p_1 \xi) + \\
+ (8b_1 p^2 \alpha + 8\Omega^2 b_2 + 32b_1 p^4) e^{\Omega y + \gamma} e^{2\Omega y + 2\gamma} = 0
\end{align*}
\]

Equating all coefficients of different powers of \( \cos(p_1 \xi) e^{2\Omega y + 2\gamma}, \) \( \cos(p_1 \xi), \) and \( e^{\Omega y + \gamma} \) to zero, we get:

\[
\begin{align*}
4b_1 p^2 \alpha b_2 + 4\Omega^2 b_2 + 4b_1 p^4 b_2 &= 0 \\
4b_1 p^2 \alpha + 4\Omega^2 b_1 + 4b_1 p^4 &= 0 \\
8b_1 p^2 \alpha + 8\Omega^2 b_2 + 32b_1 p^4 &= 0
\end{align*}
\]

(22)

Solving above equations we get:

\[ \Omega^2 = -\alpha p^2 - p^4, \quad b_2 = \frac{(\alpha + 4p^2)b_1^2}{\alpha + p^2} \]  

(23)

Substituting \( \xi = x - \alpha t \) into eq. (15), eq. (15) can be rewritten:

\[
u(x, y, t) = -\frac{-b_1 p \sin[p(x - \alpha t)]}{\sqrt{b_2 \cosh[\Omega y + \lambda] + \frac{1}{2} \ln(b_2)] + b_2 \cos[p(x - \alpha t)]}} \]  

(24)

Substituting eq. (18) into eq. (19), we get exact periodic soliton solution of eq. (1):

\[
u(x, y, t) = -\frac{-b_1 p \sin[p(x - \alpha t)]}{\sqrt{(\alpha + 4p^2)b_1^2 \cosh[p y \sqrt{-\alpha - p^2 + \gamma} + \frac{1}{2} \ln(\alpha + 4p^2)b_1^2 + b_2 \cos[p(x - \alpha t)]}}} \]  

(25)
Let $p$ tends to zero and $\gamma = 0$, $b_1 = 1$, we can get the following rational breather-wave solution:

$$u(x, y, t) = \frac{4\alpha(x - \alpha t)}{\alpha x^2 - 2\alpha \alpha' t + \alpha^2 t^2 - y^2 + 3}$$

(26)

Solutions (25) is periodic solitary wave which have speed $\alpha$, the forward-direction (or backward-direction) wave shows solitary feature meanwhile takes on periodical feature with space variable $x, t$ for PKP equation. In particular, this wave shows both solitary and periodical feature to space variable $t$. This is a strange and interesting physical phenomenon to the evolution of 2-D flow of shallow-water waves having small amplitudes. We will study this interesting phenomenon in further work. From fig. 2, it is observed that the kink of the solutions disappeared when the $p$ tends to zero. More importantly, we obtained a rational breather wave solution

Conclusions

In summary, the aid of extended homoclinic test method, we successfully applied this method to the (2+1) dimensional PKP equation. Some exact periodic soliton solution, exact breather (cross-kink) solutions and rational solutions are obtained. More importantly, it is an effective method for many equations seeking rational solutions. Our results enrich the variety of the dynamics of high-dimensional systems. This method is simple and straightforward. We will investigate other types of non-linear evolution equations and non-integrable systems.

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References


