KINK DEGENERACY AND ROGUE WAVE FOR POTENTIAL KADOMTSEV-PETVIASHVILI EQUATION

by

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A new method called homoclinic breather limit method is proposed to solve the Potential Kadomtsev-Petviashvili equation, breather kink-wave and periodic soliton are obtained, and kink degeneracy and new rogue wave are first found in this paper.

Key words: *Potential Kadomtsev-Petviashvili equation, rogue wave solution, extended homoclinic test method, kink degeneracy, exp-function method*

Introduction

In recent years, the study of exact solutions for the non-linear evolution equations has attracted much attention [1], and many non-linear phenomena were discovered [2-7].

In this work, we focus on some new non-linear phenomena, *i. e.*, the kink degeneracy and rational breather solutions, for the (2+1) dimensional Potential Kadomtsev-Petviashvili (PKP) equation:

$$u_{xt} + 6u_x u_{xx} + u_{xxxx} + u_{yy} = 0 \tag{1}$$

It is well-known that this equation arises in a number of remarkable non-linear problems in fluid mechanics and thermal science, and its solutions have been studied extensively. Solitary wave, soliton-like solution, and interaction among solitary waves were found [8-13]. Recently, exact periodic kink-wave and degenerative soliton solution of PKP were also elucidated [14]. In this paper, a novel approach named as homoclinic breather limit process is proposed to seek for rational breather-wave solutions.

Homoclinic breather limit method

In order to elucidate the new method, we consider a high dimensional non-linear evolution equation in the general form:

$$F(u,\ u_t,\ u_x,\ u_y,\ u_{xx},\ u_{yy},\ \cdots) = 0 \tag{2}$$

where $u(x, y, t)$, and F is a polynomial of u and its derivatives.

The new method takes following steps.

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Step 1. By Painleve analysis, a transformation:

$$u = T(f) \tag{3}$$

is made for a new and unknown function f.

Step 2. Convert eq. (2) into Hirota bilinear form:

$$G(D_t,\ D_x,\ D_y;\ f,\ f) = 0 \tag{4}$$

where D is the operator defined by:

$$D_x^m D_t^k a\, b = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(x,t)\, b(x',t')\Big|_{x=x',\ t=t'}$$

Step 3. Solve eq. (3) using homoclinic test approach by assuming that [15]:

$$f(x, y, t) = 1 + b_0(e^{ip(x-\alpha t)} + e^{-ip(x-\alpha t)})e^{\Omega y+\gamma} + b_1 e^{2\Omega y+2\gamma} \tag{5}$$

or by the extended homoclinic test approach by assuming that:

$$f(x, y, t) = e^{-p(\alpha x+\beta y+\omega t+\gamma)} + b_0 \cos[p_1(\alpha_1 x + \beta_1 y + \omega_1 t + \gamma_1)] + b_1 e^{p(\alpha x+\beta y+\omega t+\gamma)} \tag{6}$$

where α, β, Ω, α_1, β_1, ω_1, p, p_1, b_0, b_1, γ, and γ_1 are constants to be determined, or by a more general approach by the exp-function method [16].

Step 4. Substitute eq. (5) or eq. (6) into eq. (3), and equate all coefficients of $e^{-(\alpha x+\beta y+\omega t+\gamma)}$, $e^{\alpha x+\beta y+\omega t+\gamma}$, $\cos(\alpha_1 x + \beta_1 y + \omega_1 t + \gamma_1)$, $\sin(\alpha_1 x + \beta_1 y + \omega_1 t + \gamma_1)$, and constant term to zero, we obtain the set of algebraic equation for α, β, ω, α_1, β_1, ω_1, p, p_1, b_0, and b_1.

Step 5. Solve the set of algebraic equations in Step 4 using Maple and for α, β, Ω, α_1, β_1, ω_1, p, p_1, b_0, and b_1.

Step 6. Substitute the identified values of α, β, ω, α_1, β_1, ω_1, p, p_1, b_0, and b_1 into eq. (2) and eq. (5) and deduce the exact solutions of eq. (1).

Step 7. Using the relationship between p_1 and p, or p and Ω of exact solution obtained in the Step 6 and let p tends to zero, we can get the rational breather-wave solution.

Kink degeneracy

By using Painleve test we can assume that:

$$u(x, y, t) = 2(\ln f)_x \tag{7}$$

where $f(x, y, t)$ is an unknown real function to be determined. Substituting eq. (7) into eq. (1), we obtain the bilinear equation:

$$(D_x D_t + D_y^2 + D_x^4) f \cdot f = 0 \tag{8}$$

Now we suppose that the solution of eq. (8) is:

$$f(x, y, t) = e^{-\xi} + b_0 \cos(\eta) + b_1 e^{\xi} \tag{9}$$

where

$$\xi = p(x + \beta t + \omega), \qquad \eta = p_1(x - \beta t + \omega_1) + \alpha_1 y$$

and β, ω, α_1, ω_1, p, p_1, b_0, and b_1 are some constants to be determined later. Substituting eq. (9) into eq. (8), we have:

$$[2b_0b_1(p^4+p_1^2\beta+p^2\beta+p_1^4-\alpha_1^2-6p^2p_1^2)\cos(\eta)-8pb_0b_1p_1(p-p_1)(p+p_1)\sin(\eta)]e^{\xi}+$$
$$+[2b_0(p^4+p_1^2\beta+p^2\beta+p_1^4-\alpha_1^2-6p^2p_1^2)\cos(\eta)+8pb_0p_1(p-p_1)(p+p_1)\sin(\eta)]e^{-\xi}-$$
$$-2b_0^2\alpha_1^2+8\beta p^2b_1+8b_0^2p_1^4+2\beta b_0^2p_1^2+32p^4b_1=0$$

Equating all coefficients of different powers of e^{ξ}, $e^{-\xi}$, $\sin(\eta)$, and $\cos(\eta)$ to zero, we get:

$$\begin{cases} p^4+p_1^2\beta+p^2\beta+p_1^4-\alpha_1^2-6p^2p_1^2=0 \\ 8pb_0p_1(p-p_1)(p+p_1)=0 \\ -2b_0^2\alpha_1^2+8\beta p^2b_1+8b_0^2p_1^4+2\beta b_0^2p_1^2+32p^4b_1=0 \end{cases} \tag{10}$$

Solving the system of eqs. (10) with the aid of Maple, we get:

$$p_1=\pm p, \qquad b_1=\frac{b_0^2(\beta-8p^2)}{4(\beta+4p^2)}, \qquad \alpha_1=\pm p\sqrt{2\beta-4p^2} \tag{11}$$

and β, ω, ω_1, p, b_0, are free. Substitute eq. (11) into eq. (9), we obtain the solution:

$$f(x,y,t)=e^{-p(x+\beta t+\omega)}+b_0\cos[p_1(x-\beta t+\omega_1)+\alpha_1y]+b_1e^{p(x+\beta t+\omega)} \tag{12}$$

If $0<b_1\in R$, then we obtain the exact breather kink solution:

$$u(x,y,t)=\frac{4p\sqrt{b_1}\sinh p(x+\beta t+\omega)+\frac{1}{2}\ln(b_1)-2b_0\sin p_1(x-\beta t+\omega_1)+\alpha_1y}{2\sqrt{b_1}\cosh p(x+\beta t+\omega)+\frac{1}{2}\ln(b_1)+b_0\cos p_1(x-\beta t+\omega_1)+\alpha_1y} \tag{13}$$

Substitute eq. (11) into eq. (13), eq. (13) can be re-written as:

$$u(x,y,t)=\frac{\pm 2p\sqrt{\frac{\beta-8p^2}{\beta+4p^2}}\sinh p(x+\beta t+\omega)+\frac{1}{2}\ln\left[\frac{b_0^2(\beta-8p^2)}{4(\beta+4p^2)}\right]-2\sin p_1(x-\beta t+\omega_1)+\alpha_1y}{\pm\sqrt{\frac{\beta-8p^2}{\beta+4p^2}}\cosh p(x+\beta t+\omega)+\frac{1}{2}\ln\left[\frac{b_0^2(\beta-8p^2)}{4(\beta+4p^2)}\right]+\cos p_1(x-\beta t+\omega_1)+\alpha_1y} \tag{14}$$

Solution $u(x, y, t)$ represented by eq. (14) is a breather kink-wave solution. In fact, solution $u(x, y, t)$ is a kink wave as trajectory along the straight line $x=\beta t-\omega_1-\alpha_1y/p$, and meanwhile evolves periodically along the straight line $x=-\beta t-\omega/p$, fig. 1(a).

Let p tends to zero and take $b_0=\pm 2$, we can get the following rational breather solution:

$$u(x,y,t)=\frac{-4\beta(2x+\omega+\omega_1+\sqrt{2\beta}y)}{A+B-2\beta^3t^2+(-2\omega\beta^2+2\beta^2\omega_1)t+12-\beta\omega^2-\beta\omega_1^2}$$

where

$$\begin{cases} A=-2\beta x^2+(-2\beta\omega-2\beta\omega_1-2\beta^{3/2}\sqrt{2}y)x \\ B=-2\beta^2y^2+(2\sqrt{2}\beta^{5/2}t-2\sqrt{2}\beta^{3/2}\omega_1)y \end{cases} \tag{15}$$

The solution $u(x, y, t)$ represented by eq. (15) is a breather solution and no longer the kinky. This shows that kink is degenerated when the period of breather wave tends to infinite in the breather kink-wave, fig. 1(b). this is a new non-linear phenomenon up to now.

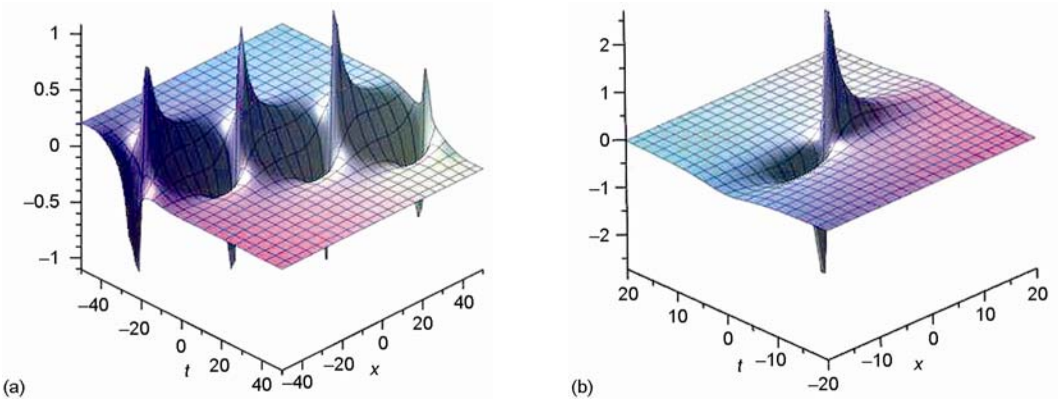

Figure 1. (a) the breather kink solution as $p = 1/10$, $\beta = -2$, $b_0 = \omega_1 = \omega = 1$, $y = 0$, and (b) the rational breather-wave solution as $\alpha = \omega_1 = \omega = 0$, $\beta = 4$, $y = 0$

Solutions (13) is kinky periodic-wave which has speed β, the forward-direction (or backward-direction) wave shows solitary feature meanwhile takes on kinky feature with space variable *x, t* for PKP equation. Specially, this wave shows both kinky and periodic feature to space variable *t*. Such a surprising feature of weakly dispersive long-wave is first obtained. Meanwhile, notice (15) when the *t* tends to infinity, *u* tends to zero. From fig. 1, it is observed that the kink of the solutions disappeared when the *p* tends to zero. More importantly, we obtained a rational breather wave solution.

From periodic soliton solution to rational breather wave solution

In this section, by choosing special test function in application of HTA to (2+1) dimensional Potential-Petviashvili equation, we obtain a periodic soliton solution and a rational breather-wave solution. Setting:

$$\xi = x - \alpha t$$

where α is a wave velocity. Equation (1) can be re-written as:

$$-\alpha u_{\xi\xi} + 6u_{\xi}u_{\xi\xi} + u_{\xi\xi\xi\xi} + u_{yy} = 0 \tag{16}$$

Using Painlev´e analysis, we suppose that the solution of eq. (11) is:

$$u(x, y, t) = 2(\ln f)_{\xi} \tag{17}$$

for some unknown real function *f*(*x, y, t*) and by substituting eq. (11) into eq. (12), we can obtain the bilinear form:

$$(-\alpha D_{\xi}^{2} + D_{y}^{2} + D_{\xi}^{4}) f \cdot f = 0 \tag{18}$$

With regard to eq. (13), using the homoclinic test technique [14], we are going to seek the solution of the form:

$$f(\xi, y) = 1 + b_1(\mathrm{e}^{ip\xi} + \mathrm{e}^{-ip\xi})\mathrm{e}^{\Omega y+\gamma} + b_2\mathrm{e}^{2\Omega y+2\gamma} \tag{19}$$

where p, Ω, γ, b_1, and b_2 are all real to be determined below. Substituting eq. (14) into eq. (13) yields the exact solution of eq. (12) in the form:

$$u(x, y, t) = \frac{-2b_1 p\mathrm{e}^{\Omega y+\lambda}\ \sin(p\xi)}{1 + 2b_1\ \cos(p\xi)\mathrm{e}^{\Omega y+\gamma} + b_2\mathrm{e}^{2(\Omega y+\gamma)}} \tag{20}$$

Computing $D_\xi^2 f \cdot f$, $D_y^2 f \cdot f$, and $D_\xi^4 f \cdot f$, we obtain:

$$\begin{cases} \mathrm{D}_\xi^2 f \cdot f = -4b_1 p^2 \mathrm{e}^{\Omega y+\gamma}[\cos(p\xi)\mathrm{e}^{2\Omega y+2\gamma}b_2 + \cos(p\xi) + 2b_1\mathrm{e}^{\Omega y+\lambda}] \\ \mathrm{D}_y^2 f \cdot f = 4\Omega^2 e^{\Omega y+\gamma}[\cos(p\xi)b_1\mathrm{e}^{2\Omega y+2\gamma}b_2 + \cos(p\xi)b_1 + 2b_2\mathrm{e}^{\Omega y+\lambda}] \\ \mathrm{D}_\xi^4 f \cdot f = 4b_1 p^4 e^{\Omega y+\gamma}[\cos(p\xi)\mathrm{e}^{2\Omega y+2\gamma}b_2 + \cos(p\xi) + 8b_1\mathrm{e}^{\Omega y+\lambda}] \end{cases} \tag{21}$$

Substituting eqs. (16) into eq. (13) we get:

$$[(4b_1 p^2\alpha b_2 + 4\Omega^2 b_1 b_2 + 4b_1 p^4 b_2)\cos(p\xi)\mathrm{e}^{2\Omega y+2\gamma} + (4b_1 p^2\alpha + 4\Omega^2 b_1 + 4b_1 p^4)\cos(p\xi) +$$

$$+\ (8b_1^2 p^2\alpha + 8\Omega^2 b_2 + 32b_1^2 p^4)\mathrm{e}^{\Omega y+\gamma}]\mathrm{e}^{\Omega y+\gamma} = 0$$

Equating all coefficients of different powers of $\cos(p\xi)\mathrm{e}^{2\Omega y+2\gamma}$, $\cos(p\xi)$, and $\mathrm{e}^{\Omega y+\gamma}$ to zero, we get:

$$\begin{cases} 4b_1 p^2\alpha b_2 + 4\Omega^2 b_1 b_2 + 4b_1 p^4 b_2 = 0 \\ 4b_1 p^2\alpha + 4\Omega^2 b_1 + 4b_1 p^4 = 0 \\ 8b_1^2 p^2\alpha + 8\Omega^2 b_2 + 32b_1^2 p^4 = 0 \end{cases} \tag{22}$$

Solving above equations we get:

$$\Omega^2 = -\alpha p^2 - p^4, \qquad b_2 = \frac{(\alpha + 4p^2)b_1^2}{\alpha + p^2} \tag{23}$$

Substituting $\xi = x - \alpha t$ into eq. (15), eq. (15) can be rewritten:

$$u(x, y, t) = \frac{-b_1 p \sin[p(x - \alpha t)]}{\sqrt{b_2}\ \cosh[(\Omega y + \lambda) + \frac{1}{2}\ln(b_2)] + b_1\ \cos[p(x - \alpha t)]} \tag{24}$$

Substituting eq. (18) into eq. (19), we get exact periodic soliton solution of eq. (1):

$$u(x, y, t) = \frac{-b_1 p \sin[p(x - \alpha t)]}{\sqrt{\frac{(\alpha + 4p^2)b_1^2}{\alpha + p^2}}\cosh\left[py\sqrt{-\alpha - p^2} + \gamma + \frac{1}{2}\ln\frac{(\alpha + 4p^2)b_1^2}{\alpha + p^2}\right] + b_1\ \cos[p(x - \alpha t)]} \tag{25}$$

Let p tends to zero and $\gamma = 0$, $b_1 = 1$, we can get the following rational breather-wave solution:

$$u(x, y, t) = \frac{4\alpha(x - \alpha t)}{\alpha x^2 - 2x\alpha^2 t + \alpha^3 t^2 - y^2\alpha^2 + 3} \tag{26}$$

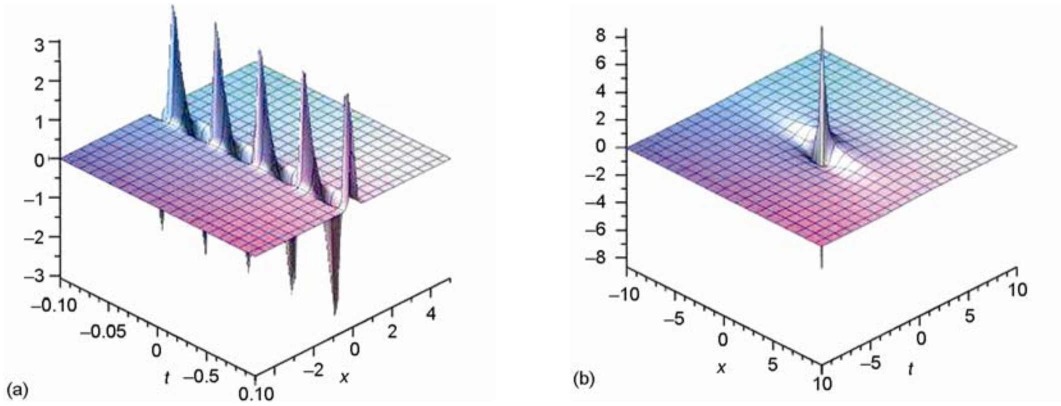

Figure 2. (a) the periodic soliton solution as $p = b_1 = \sqrt{2}$, $\alpha = -100$, $\gamma = 0$, $y = x$, and (b) the rational breather-wave solution as $\alpha = -6$, $y = x$

Solutions (25) is periodic solitary wave which have speed α, the forward-direction (or backward-direction) wave shows solitary feature meanwhile takes on periodical feature with space variable *x, t* for PKP equation. In particular, this wave shows both solitary and periodic feature to space variable *t*. This is a strange and interesting physical phenomenon to the evolution of 2-D flow of shallow-water waves having small amplitudes. We will study this interesting phenomenon in further work. From fig. 2, it is observed that the kink of the solutions disappeared when the *p* tends to zero. More importantly, we obtained a rational breather wave solution

Conclusions

In summary, the aid of extended homoclinic test method, we successfully applied this method to the (2+1) dimensional PKP equation. Some exact periodic soliton solution, exact breather cross-kink solutions and rational solutions are obtained. More importantly, it is an effective method for many equations seeking rational solutions. Our results enrich the variety of the dynamics of high-dimensional systems. This method is simple and straightforward. We will investigate other types of non-linear evolution equations and non-integrable systems.

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