PAINLEVE ANALYSIS FOR A FORCED KORTEVEG-DE VRIES EQUATION ARISEN IN FLUID DYNAMICS OF INTERNAL SOLITARY WAVES

by

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In this paper, Painleve analysis is used to test the Painleve integrability of a forced variable-coefficient extended Korteveg-de Vries equation which can describe the weakly-non-linear long internal solitary waves in the fluid with continuous stratification on density. The obtained results show that the equation is integrable under certain conditions. By virtue of the truncated Painleve expansion, a pair of new exact solutions to the equation is obtained.

Key words: *Painleve analysis; forced Korteveg-de Vries equation; exact solution; Weiss-Tabor-Carnevale method*

Introduction

Non-linear evolution equations (NLEE) are usually related to some non-linear physical phenomena, for example, the celebrated Korteweg-de Vries (KdV) equation is a prototype model to govern some wave phenomena in the atmosphere, plasma, astrophysics, and transmission liners [1, 2]. Besides, the KdV equation can describe the internal solitary waves (ISW) in shallow water. As a type of non-linear wave in stratified fluid with continuous stratification on the density or current, the ISW are frequently observed close to the oceanic regions of steep topography such as the shelf-edges, ridges, and sills [3]. Since 1834 when the soliton phenomena was first observed and the KdV equation was solved by the inverse scattering method [4], many exact solutions of NLEE have been obtained [3-8]. Testing Painleve integrability of NLEE plays an important role in finding exact solutions of NLEE, and has become one of the most significant tasks in soliton theory. The present paper is motivated by the desire to use the Weiss-Tabor-Carnevale (WTC) method [9] of Painleve analysis to prove the following forced variable-coefficient extended KdV equation [3]:

$$u_t + a(t)uu_x + b(t)u^2u_x + c(t)u_{xxx} + d(t)u_x + \gamma(t)u = \Gamma(t) \tag{1}$$

has the Painleve integrability under certain conditions and hence construct its exact solutions, here the variable coefficients $a(t)$, $b(t)$, $c(t)$, … $\gamma(t)$, and $\Gamma(t)$ are smooth functions of t. If the external force term $\Gamma(t)$ vanishes, eq. (1) can describe the weakly non-linear long ISW in the fluid with continuous stratification on the density. Liu *et al*. [3], utilized Hirota's bi-linear method to solve eq. (1) and obtained multi-soliton solutions under the constraint of

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$\gamma(t) = 0$. In this paper, we shall employ the truncated Painleve expansion derived in the next section to construct new exact solutions without such a constraint.

Painleve property

We take a transformation:

$$u = v + \int \Gamma(t)\,\mathrm{d}t, \qquad v = v(x,t) \tag{2}$$

then eq. (1) is converted into:

$$v_t + [a(t) + 2b(t)\int \Gamma(t)\mathrm{d}t]vv_x + \{a(t)\int \Gamma(t)\mathrm{d}t + b(t)[\int \Gamma(t)\mathrm{d}t]^2 + d(t)\}v_x +$$

$$+b(t)v^2 v_x + c(t)v_{xxx} + \gamma(t)v + \gamma(t)\int \Gamma(t)\mathrm{d}t = 0 \tag{3}$$

Further supposing:

$$v = \varphi^{-\rho}(x,t)\sum_{j=0}^{\infty} v_j(x,t)\varphi^j(x,t), \qquad \rho > 0 \tag{4}$$

and substituting eq. (4) into eq. (3), we have:

$$\rho = 1, \quad v_0 = \pm\left[-\frac{6c(t)}{b(t)}\right]^{\frac{1}{2}} \varphi_x \tag{5}$$

Substituting eq. (4) along with eq. (5) into eq. (3), and setting each coefficient of φ^{-3}, φ^{-2}, and φ^{j-4} to zero yields:

$$v_1 = \mp \frac{\varphi_{xx}}{2\varphi_x}\left[-\frac{6c(t)}{b(t)}\right]^{\frac{1}{2}} - \frac{a(t)}{2b(t)} - \int \Gamma(t)\mathrm{d}t \tag{6}$$

$$v_2 = \pm\left[-\frac{6c(t)}{b(t)}\right]^{\frac{1}{2}}\left[\frac{\varphi_t}{6c(t)\varphi_x^2} - \frac{a^2(t)}{24b(t)c(t)\varphi_x} + \frac{d(t)}{6c(t)\varphi_x} - \frac{\varphi_{xx}^2}{4\varphi_x^3} + \frac{\varphi_{xxx}}{6\varphi_x^2}\right] \tag{7}$$

$$-(j+1)(j-3)(j-4)c(t)v_j\varphi_x^3 = v_{j-3,t} + (j-3)v_{j-2}\varphi_t + [a(t) + 2b(t)\int \Gamma(t)\mathrm{d}t]\cdot$$

$$\cdot\left[\sum_{k=0}^{j-2} v_k v_{j-2-k} + \sum_{k=0}^{j-1}(j-2-k)v_k v_{j-1-k}\varphi_x\right] + \{a(t)\int \Gamma(t)\mathrm{d}t + b(t)[\int \Gamma(t)\mathrm{d}t]^2 + d(t)\}\cdot$$

$$\cdot[v_{j-3.x} + (j-3)v_{j-2.x}\varphi_x] + b(t)\left[\sum_{m=0}^{j-1}\sum_{p=0}^{j-1} v_{m,x}v_p v_{j-1-p-m} + \sum_{m=1}^{j-1}\sum_{p=1}^{j-1}(m-1)v_m v_p v_{j-m-p}\right] +$$

$$+ c(t)[v_{j-3,xxx} + 3(j-3)v_{j-2,xx}\varphi_x + 3(j-2)(j-3)v_{j-1,x}\varphi_x^2 + 3(j-3)v_{j-2,x}\varphi_{xx} +$$

$$+ 3(j-2)(j-3)v_{j-1}\varphi_x\varphi_{xx} + (j-3)v_{j-2}\varphi_{xxx}] + \gamma(t)v_{j-3} \tag{8}$$

We therefore obtain resonance points $j = -1, 3, 4$, among them $j = -1$ corresponds to the arbitrariness of singular manifold. Thus, all v_i in eq. (4) can be determined from eqs. (6)-(8) except for v_3 and v_4. Setting $j = 3, 4$, with the help of eqs. (5)-(7) we can derive the following constrains from the compatibility condition (8):

$$c'(t) = c(t)\left[-2\gamma(t) + \frac{b'(t)}{b(t)}\right], \qquad a'(t) = -a(t)\gamma(t) - 2b(t)\Gamma(t) + \frac{a(t)b'(t)}{b(t)} \tag{9}$$

It is easy to see that if eq. (1) possesses Painleve property then eq. (9) must hold. In other words, eq. (9) gives the conditions that eq. (1) is fully integrable in the sense of WTC method.

Exact solutions

In order to construct exact solutions of eq. (1), we set $v_2 = v_3 = v_4 = 0$, then eq. (8) gives $v_5 = v_6 = \ldots = 0$. In this case, eq. (4) is truncated:

$$v = \pm\left[-\frac{6c(t)}{b(t)}\right]^{\frac{1}{2}}\left[\varphi_x\varphi^{-1} - \frac{\varphi_{xx}}{2\varphi_x}\right] - \frac{a(t)}{2b(t)} - \int \Gamma(t)\mathrm{d}t \tag{10}$$

where φ satisfies the condition:

$$\varphi_t - \frac{a^2(t)\varphi_x}{4b(t)} + d(t)\varphi_x - \frac{3c(t)\varphi_{xx}^2}{2\varphi_x} + c(t)\varphi_{xxx} = 0 \tag{11}$$

Selecting $\varphi = 1 + \exp[kx + \int w(t)\mathrm{d}t]$ and substituting it into eq. (11), we have:

$$w(t) = -kd(t) + \frac{k^3c(t)}{2} + \frac{ka^2(t)}{4b(t)} \tag{12}$$

and hence obtain a pair of new exact solutions of eq. (1):

$$u = -\frac{a(t)}{2b(t)} \pm \frac{k}{2}\left[-\frac{6c(t)}{b(t)}\right]^{\frac{1}{2}} \frac{\mathrm{e}^{kx+\int w(t)\mathrm{d}t} - 1}{\mathrm{e}^{kx+\int w(t)\mathrm{d}t} + 1} \tag{13}$$

where $w(t)$ is determined by eq. (12), while $a(t)$, $b(t)$, and $c(t)$ satisfy two constrains in eq. (9). In fig. 1, a local structure of the kink-type solutions eq. (13) is shown.

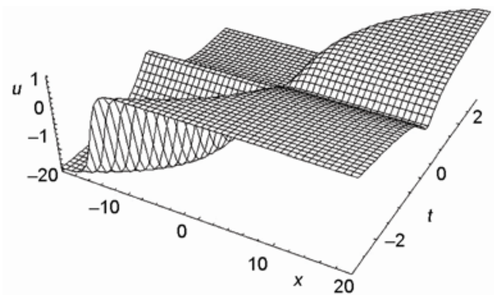

Figure 1. Local structure of kink-type solutions eq. (13) with (+) branch for parameters $a(t) = t\sin(t)$, $b(t) = t^2 + 1$, $d(t) = 1 + 0.5t\tanh(t)$, $k = 1.5$, $w(t) = 0.375[t^2\sin^2(t)/(1 + t^2) - 4.5t^2 - 4 - 2t\tanh(t)]$, $\gamma(t) = t/(1 + t^2) - 1/t$, and $\Gamma(t) = [0.5\sin(t) - t]/(1 + t^2) + 0.5t^2\sin(t)/(1 + t^2)^2$

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