# IMPROVED HYPERBOLIC FUNCTION METHOD AND EXACT SOLUTIONS FOR VARIABLE COEFFICIENT BENJAMIN-BONA-MAHONY-BURGERS EQUATION

by

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*By using the improved hyperbolic function method, we investigate the variable coefficient Benjamin-Bona-Mahony-Burgers equation which is very important in fluid mechanics. Some exact solutions are obtained. Under some conditions, the periodic wave leads to the kink-like wave.*

Key words: *Benjamin-Bona-Mahony-Burgers equation, exact solutions, hyperbolic function method*

## Introduction

Nowadays, non-linear partial differential equations (NLPDE) are becoming more and more important, especially their broad applications on modeling many physical phenomena. For instance, solid state physics, fluid mechanics, and so on [1]. It is very valuable to do some research on how to solve NLPDE exact solutions and investigate the property of the solutions.

The Benjamin-Bona-Mahony-Burgers (BBMB) equation:

$$u_t - u_{xxt} - \alpha u_{xx} + uu_x + u_x = 0$$

is proposed as a model to study the unidirectional long waves of small amplitudes in water, which is an alternative to the KdV equation [2]. It is an important model in fluid mechanics.

In this paper we will study the variable coefficient BBMB equation:

$$u_t - \lambda(t)u_{xxt} - \alpha(t)u_{xx} + \beta(t)uu_x + \gamma(t)u_x = 0 \tag{1}$$

where $\lambda(t), \alpha(t), \beta(t),$ and $\gamma(t)$ are arbitrary time-dependent coefficients.

If $\alpha(t) = 0$, $\lambda(t) = 1$, and $\gamma(t) = 1$, eq. (1) is an alternative to the regularized long wave equation proposed by Benjamin *et al.* [3] and Peregrine [4]. The variable coefficient BBMB eq. (1) [5] is proposed to describe long waves of small amplitudes broadcasting in non-linear dispersive media. This model has been introduced [6], which plays a very important role in mathematics and fluid. During the past few years, numerous methods in solving BBMB equations have been discovered by many authors, such as *G′/G* method, homotopy analysis method, He' method, *etc.* [7, 8].

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Hyperbolic function method was first proposed in [9], which is based on the fact that most solitary wave solutions have form of hyperbolic function. This method has been used to find the exact solutions of many non-linear wave equations, such as Burgers' equation, KdV equation, *etc.* [9, 10].

## Improved hyperbolic function method

For partial differential equation (PDE) of two independent variables $x$, $t$:

$$\mathrm{P}\,(t,\ x,\ u_x,\ u_t,\ u_{xx}, ...) = 0 \tag{2}$$

we consider its traveling wave solution of the form:

$$u(x, t) = \varphi(\xi), \qquad \xi = k(x) + c(\mathrm{t}) + l \tag{3}$$

where $k(x)$ and $c(t)$ are functions of single variable.

By using this traveling wave transaction, we can change non-linear PDE eq. (2) to a variable coefficient differential equation (VCDE).

Supposing that the solution of this VCDE is:

$$\phi(\xi) = \sum_{i=0}^{m} a_i f^i + \sum_{j=1}^{m} b_j f^{j-1} g \tag{4}$$

where

$$f = \frac{1}{\cosh\,\xi + r}, \qquad g = \frac{\sinh\,\xi}{\cosh\,\xi + r} \tag{5}$$

and $a_i$, $b_j$, and $r$ are coefficients to be determined. $m$ can be determined by balancing [11] the highest order derivatives and highest order non-linear terms appearing in VCDE, what's more:

$$f_\xi = -fg, \qquad g_\xi = 1 - g^2 - rf, \qquad g^2 = 1 - 2rf + (r^2 - 1) f^2 \tag{6}$$

By substituting eq. (4) into eq. (2), and using eq. (5) to simplify the VCDE until it only involves exponential term of $f$ and $g$ (the degree of $g$ is less than 1). Then combining the similar terms of $f$ and $g$, we will establish a set of equation. By solving this set of equation, the exact solution of eq. (2) will be obtained.

## Obtain exact solution of BBMB equation by improved hyperbolic function method

Firstly, we consider the following transformations:

$$u(x, t) = \varphi(\xi), \qquad \xi = k(x) + c(t) + l \tag{7}$$

where $k(x)$, $c(t)$ are arbitrary functions.

Substituting eq. (7) into eq. (1) we have the equation:

$$[c'(t) - \alpha(t)k''(x) + \gamma(\mathrm{t})]\varphi'(\xi) - [\lambda(t)c'(t)k''(x) + \alpha(t)k^{2'}(x)]\varphi''(\xi) +$$

$$+\beta(t)k'(x)\varphi(\xi)'(\xi) - \lambda(t)c'(t)k^{2'}(x)\varphi'''(\xi) = 0 \tag{8}$$

By using eq. (4) and balancing the highest order derivatives and highest order non-linear terms appearing in eq. (8), we get $m = 1$. Thus we have:

$$\phi(\xi) = a_0 + a_1 f + b_1 g \tag{9}$$

By substituting eq. (9) into eq. (8), simplifying the eq. (8) until it only involves exponential term of $f$ and $g$ (the degree of $g$ is less than 1), by supposing their coefficient equal to zero, we obtain the system:

$$\begin{cases}
(1)\quad b_1(r+1)(r-1)\lambda(t)c'(t)k^{2'}(x) = 0 \\
(2)\quad (r+1)(r-1)\{a_1\lambda(t)c'(t)k''"(x) + a_1 b_1(t)k'(x) + [a_1\alpha(t) + 12b_1 r\lambda(t)c'(t)]k^{2'}(x)\} = 0 \\
(3)\quad [3a_1 r\lambda(t)c'(t) - b_1(1-r^2)\alpha(t)]k''(x) + b_1[(3a_1 r + a_0 - a_0 r^2)\beta(t) + (1-r^2)\gamma(t)]k'(x) + \\
\quad +\ [3a_1 r\alpha(t) + b_1(7r^2-4)\lambda(t)c'(t)]k^{2'}(x) + b_1(1-r^2)c'(t) = 0 \\
(4)\quad [b_1 rc'(t) - b_1 r\alpha(t) - a_1\lambda(t)c'(t)]k''(x) + [a_0 r b_1\beta(t) - a_1 b_1\beta(t) + b_1 r\gamma(t)]k'(x) - \\
\quad -\ [a_1\alpha(t) + b_1 r\lambda(t)c'(t)]k^{2'}(x) + b_1 rc'(t) = 0 \\
(5)\quad [b_1 r\lambda(t)c'(t) + a_1\alpha(t)]k''(x) + [(b_1^2 r - a_0 a_1)\beta(t) - a_1\gamma(t)]k'(x) + \\
\quad +\ [a_1\gamma(t)c'(t) + b_1 r\alpha(t)]k^{2'}(x) - a_1 c'(t) = 0 \\
(6)\quad 2b_1(r^2-1)\lambda(t)c'(t)k''(x) - (b_1^2 - b_1^2 r^2 - a_1^2)\beta(t)k'(x) + \\
\quad +\ [2b_1(r^2-1)\alpha(t) + 6a_1 r\lambda(t)c'(t)]k^{2'}(x) = 0 \\
(7)\quad a_1(r+1)(r-1)\lambda(t)c'(t)k^{2'}(x) = 0
\end{cases}$$

Solving this system by Maple gives the following results:
(1) When $\lambda(t) = 0$, we obtain:

$$k'(x) = \frac{-b_1\beta(t)}{\alpha(t)} \tag{10}$$

$$a_1 = \pm\sqrt{(r^2-1)}\ b_1 \tag{11}$$

$$c'(t) = \frac{b_1\gamma(t)\beta(t) + a_0 b_1\beta^2(t)}{\alpha(t)} \tag{12}$$

Equation (10) indicates that $\beta(t)/\alpha(t)$ must be a constant:

$$\beta(t) = c_1\alpha(t), \qquad k'(x) = -b_1 c_1, \qquad c'(t) = b_1 c_1[\gamma(t) + a_0\beta(t)]$$

We obtain the solution of variable coefficient BBMB eq. (1):

$$u(x,t) = a_0 \pm \frac{b_1\sqrt{r^2-1}}{r + \cosh[k(x) + c(t) + l]} + \frac{b_1 \sinh[k(x) + c(t) + l]}{r + \cosh[k(x) + c(t) + l]} \tag{13}$$

where

$$k(x) = -b_1 c_1 x, \qquad c(t) = b_1 c_1 \int [\gamma(t) + a_0\beta(t)]\mathrm{d}t$$

and $a_0$, $b_1$, and $c_1$ are constants.

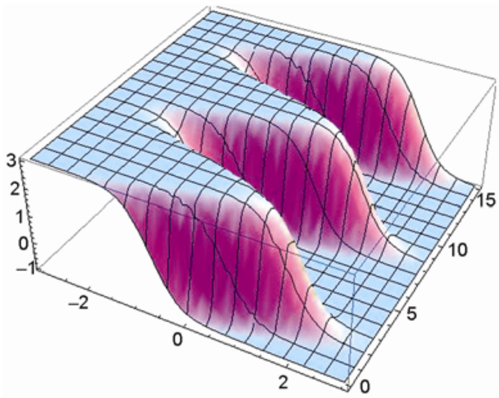


**Figure 1.** $a_0 = 1$, $b_1 = c_1 = r = 2$, $l = 0$, $\gamma(t) = \cos t$, **and** $\beta(t) = \sin t$

Figure 1 shows a special solution of eq. (1), when we suppose that:

$$a_0 = 1, \qquad b_1 = c_1 = r = 2, \qquad l = 0,$$
$$\gamma(t) = \cos t, \qquad \beta(t) = \sin t$$

From the figure we can see that the wave is a kind of kink-like wave and the amplitude is constant in the process of transmission. Since $\gamma(t)$ and $\beta(t)$ are time dependent periodic function, wave velocity (both size and direction) periodically changes over time.

(2) When $a_1 = r = 0$, we have:

$$k'(x) = -\frac{b_1\beta(t)}{2\alpha(t)} \tag{14}$$

$$c'(t) = \frac{b_1\gamma(t)\beta(t) + a_0 b_1 \beta^2(t)}{2\alpha(t)} \tag{15}$$

Equation (14) indicates that $\beta(t)/\alpha(t)$ must be a constant.
Then, we have:

$$k(x) = -\frac{b_1}{2}c_2 x, \qquad c(t) = \frac{b_1}{2}c_2 \int [\gamma(t) + a_0\beta(t)]\mathrm{d}t$$

The solution of variable coefficient BBMB equation eq. (1) is:

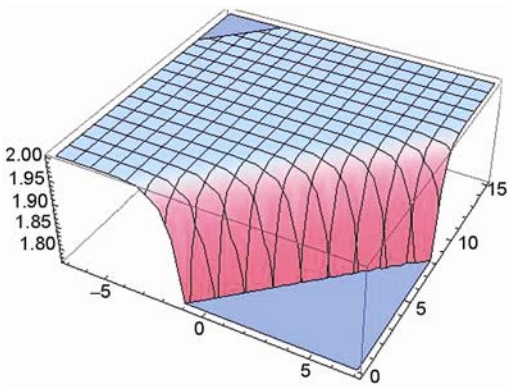


**Figure 2.** $a_0 = b_1 = 1$, $c_2 = 2$, $l = 0$, $\gamma(t) = \cos t$, **and** $\beta(t) = \sin t$

$$u(x, t) = a_0 + \frac{b_1 \sinh[k(x) + c(t) + l]}{r + \cosh[k(x) + c(t) + l]} \tag{16}$$

where

$$k(x) = -\frac{b_1}{2}c_2 x$$

$$c(t) = \frac{b_1}{2}c_2 \int [\gamma(t) + a_0\beta(t)]\mathrm{d}t$$

and $a_0$, $b_1$, and $c_2$ are constants.

Figure 2 shows a special solution when we suppose that:

$$a_0 = b_1 = 1, \qquad c_2 = 2, \qquad l = 0,$$
$$\gamma(t) = \cos t, \qquad \beta(t) = \sin t$$

From the figure we can see that the amplitude of the wave tend to be a constant over time in the process of transmission.

## Conclusions

In this article, the improved hyperbolic function method is utilized to find the solutions of the variable coefficient BBMB equation. It is very important in fluid mechanics. As we know, hyperbolic function method is an effective method in solving non-linear partial differential equations. We also indicate that hyperbolic function method is a strong method not only in solving constant coefficient non-linear partial differential equations but also in solving variable coefficient non-linear partial differential equations.

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